

ON THE COHERENCE CONJECTURE OF PAPPAS AND RAPOPORT

XINWEN ZHU

ABSTRACT. We prove the (generalized) coherence conjecture of Pappas and Rapoport proposed in [PR3]. As a corollary, one of the main theorems in [PR4], which describes the geometry of the special fibers of the local models for ramified unitary groups, holds unconditionally. Our proof is based on the study of the geometry (in particular certain line bundles and ℓ -adic sheaves) of the global Schubert varieties, which are the (generalized) equal characteristic counterparts of the local models.

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1. INTRODUCTION

The goal of this paper is to prove the coherence conjecture of Pappas and Rapoport as proposed in [PR3]. The precise formulation of the conjecture is a little bit technical and will be given in §2.3. In this introduction, we would like to describe a vague form of this conjecture, to convey the ideas behind it and to outline the proofs.

The coherence conjecture was proposed by Pappas and Rapoport in order to understand the special fibers of local models. Local models were systematically introduced by Rapoport and Zink in [RZ] (special cases were constructed earlier by Deligne-Pappas [DP] and independently by de Jong [dJ]) as a tool to analyze the étale local structure of certain integral models of (PEL-type) Shimura varieties with parahoric level structures over p -adic fields. Unlike the Shimura varieties themselves, which are usually moduli of abelian varieties, local models are defined in terms of linear algebras and therefore are much easier to study. For example, using local models, Görtz (see [Go1, Go2]) proved the flatness of certain PEL-type Shimura varieties associated to unramified unitary groups and symplectic groups (some special cases were obtained in earlier works [CN, dJ, DP]). On the other hand, a discovery of G. Pappas (cf. [Pa]) showed that the originally defined integral models in [RZ] are usually not flat when the groups are ramified. Therefore, nowadays the (local) models defined in [RZ] are usually called the naive models. In a series of papers ([PR1, PR2, PR4]), Pappas and Rapoport investigated the corrected definition of flat local models. The easiest definition of these local models is by taking the flat closures of the generic fibers in the naive local models. Usually, an integral model defined in this way is not useful since the moduli interpretation is lost and therefore it is very difficult to study the special fiber, etc (in fact a considerable part [PR1, PR2, PR4] is devoted in an attempt to cutting out the correct closed subschemes inside the naive models by strengthening the original moduli problem of [RZ]). Indeed, most investigations of local models so far used these strengthened moduli problems in a way or another (for a survey of most progresses in this area, we refer to the recent paper [PRS]).

However, as observed by Pappas and Rapoport in [PR3], the brutal force definition of the local models by taking the flat closures is not totally out of control as one may think. Namely, it is well-known that the special fibers of the naive models always embed in the affine flag varieties and that their reduced subschemes are union of Schubert varieties. Therefore, it is a question to describe which Schubert varieties will appear in the special fibers (of the flat models) and whether the special fibers are reduced. These questions are reduced to the coherence conjecture (see [PR3, PR4], at least in the case the group splits over a tamely ramified extension), which characterizes the dimension of the spaces of global sections of certain ample line bundles on certain union of Schubert varieties. Therefore, we will have a fairly good understanding of the local models even if we do not know the moduli problem they represent, provided we can prove the coherence conjecture.

Let us be a little bit more precise. To this goal, we need first recall the theory of affine flag varieties (we refer to §2.2 for unexplained notations and more details). Let k be a field and \mathcal{G} be a flat affine group scheme of finite type over $k[[t]]$. Let G be fiber of \mathcal{G} over the generic point $F = k((t)) = k[[t]][t^{-1}]$. Then one can define the affine flag variety $\mathcal{F}\ell_{\mathcal{G}} = LG/L^+\mathcal{G}$, which is an ind-scheme, of ind-finite type (cf. [BL1, Fa, PR3] and §2.2). When G is an almost simple, simply-connected algebraic group over $k((t))$, and \mathcal{G} is a parahoric group scheme of G , $\mathcal{F}\ell_{\mathcal{G}}$ is ind-projective and coincides with the affine flag varieties arising from the theory of affine Kac-Moody groups as developed in [Ku, Ma] (at least when G splits over a tamely ramified

extension of $k((t))$). The jet group $L^+\mathcal{G}$ acts on $\mathcal{F}\ell_{\mathcal{G}}$ by left translations and the orbits are finite dimensional, whose closures are called Schubert varieties. When \mathcal{G} is an Iwahori group scheme of G , Schubert varieties are parameterized by elements in the affine Weyl group W_{aff} of G (more generally, if G is not simply-connected, they are parameterized by elements in the Iwahori-Weyl group \widetilde{W}). For $w \in \widetilde{W}$, we denote the corresponding Schubert variety by $\mathcal{F}\ell_w$.

Let us come back to local models. Let $(G, K, \{\mu\})$ be a triple, where G is a reductive group over a p -adic field F , with finite residue field k_F , K is a parahoric subgroup of G and $\{\mu\}$ is a geometric conjugacy class of one-parameter subgroups of G . Let E/F be the reflex field (i.e. the field of definition for $\{\mu\}$), with ring of integers \mathcal{O}_E and residue field k_E . Then for most such triples (at least when μ is minuscule, cf. [PRS] for a complete list), one can define the so-called naive model $\mathcal{M}_{K, \{\mu\}}^{\text{naive}}$, which is an \mathcal{O}_E -scheme, whose generic fiber is the flag variety $X(\mu)$ of parabolic subgroups of G_E of type μ . Inside $\mathcal{M}_{K, \{\mu\}}^{\text{naive}}$, one defines $\mathcal{M}_{K, \{\mu\}}^{\text{loc}}$ as the flat closure of the generic fiber (for an example of the definitions of such schemes, see §8). In all known cases, one can find a reductive group G' defined over $k((t))$ and a parahoric group scheme \mathcal{G} over $k[[t]]$, such that the special fiber

$$\overline{\mathcal{M}}_{K, \{\mu\}}^{\text{naive}} := \mathcal{M}_{K, \{\mu\}}^{\text{naive}} \otimes k_E$$

embeds into the affine flag variety $\mathcal{F}\ell_{\mathcal{G}} = LG'/L^+\mathcal{G}$ as a closed subscheme, which is in addition invariant under the action of $L^+\mathcal{G}$. In particular, the reduced subscheme of $\overline{\mathcal{M}}_{K, \{\mu\}}^{\text{naive}}$ is union of Schubert varieties inside $\mathcal{F}\ell_{\mathcal{G}}$. Which Schubert variety will appear in $\overline{\mathcal{M}}_{K, \{\mu\}}^{\text{naive}}$ usually can be read from the moduli definition of $\mathcal{M}_{K, \{\mu\}}^{\text{naive}}$. However, the special fiber of $\mathcal{M}_{K, \{\mu\}}^{\text{loc}}$ is more mysterious, and a lot of work has been done in order to understand it (we refer to [PRS] (in particular its Section 4) and references therein for a detailed survey of the current progress).

Here we review two strategies to study $\mathcal{M}_{K, \{\mu\}}^{\text{loc}}$. For simplicity, we assume that the derived group of G is simply-connected and K is an Iwahori subgroup of G at this moment. In this case, \mathcal{G} will be an Iwahori group scheme of G' . One can attach $\{\mu\}$ a subset $\text{Adm}(\mu)$ in the Iwahori-Weyl group \widetilde{W} , usually called the μ -admissible set (cf. [R2] and §2.1 for the definitions). In all known cases, it is not hard to see that the Schubert varieties $\mathcal{F}\ell_w$ for $w \in \text{Adm}(\mu)$ indeed appear in $\overline{\mathcal{M}}_{K, \{\mu\}}^{\text{loc}}$, i.e.

$$\mathcal{A}(\mu) := \bigcup_{w \in \text{Adm}(\mu)} \mathcal{F}\ell_w \subset \overline{\mathcal{M}}_{K, \{\mu\}}^{\text{loc}}.$$

Now, the first strategy to determine the (underlying reduced closed subscheme of) the special fiber $\overline{\mathcal{M}}_{K, \{\mu\}}^{\text{loc}}$ goes as follows. Write down a moduli functor $\mathcal{M}'_{K, \{\mu\}}$ which is a closed subscheme of $\mathcal{M}_{K, \{\mu\}}^{\text{naive}}$, such that

$$\mathcal{M}'_{K, \{\mu\}} \otimes E = \mathcal{M}_{K, \{\mu\}}^{\text{naive}} \otimes E, \quad \overline{\mathcal{M}}'_{K, \{\mu\}}(\bar{k}) = \mathcal{A}(\mu)(\bar{k}),$$

where \bar{k} is an algebraic closure of k_E . Clearly, this will imply that the reduced subscheme

$$(1.0.1) \quad (\overline{\mathcal{M}}_{K, \{\mu\}}^{\text{loc}})_{\text{red}} = \mathcal{A}(\mu).$$

In fact, almost all the previous works to describe $\mathcal{M}_{K, \{\mu\}}^{\text{loc}}$ followed this strategy. However, let us mention that (so far) the definitions of $\mathcal{M}'_{K, \{\mu\}}$ itself is not group theoretical (i.e. it depends on choosing some representations of the group G). In

particular, when G is ramified, its definition can be complicated. In addition, except a few cases, it is not known whether $\mathcal{M}'_{K,\{\mu\}} = \mathcal{M}^{\text{loc}}_{K,\{\mu\}}$ in general.

There is another strategy to determine $\overline{\mathcal{M}}^{\text{loc}}_{K,\{\mu\}}$, as proposed in [PR3]. Namely, let us pick up an ample line bundle \mathcal{L} over $\mathcal{M}^{\text{navie}}_{K,\{\mu\}}$. Then since by definition $\mathcal{M}^{\text{loc}}_{K,\{\mu\}}$ is flat over \mathcal{O}_E with generic fiber $X(\mu)$, for $n \gg 0$,

$$\dim_{k_E} \Gamma(\mathcal{A}(\mu), \mathcal{L}^n) \leq \dim_{k_E} \Gamma(\overline{\mathcal{M}}^{\text{loc}}_{K,\{\mu\}}, \mathcal{L}^n) = \dim_E \Gamma(X(\mu), \mathcal{L}^n).$$

The general expectation (which has been verified in all known cases) that

$$\overline{\mathcal{M}}^{\text{loc}}_{K,\{\mu\}} = \mathcal{A}(\mu)$$

led Pappas and Rapoport to conjecture the following equivalent statement

$$\dim_{k_E} \Gamma(\mathcal{A}(\mu), \mathcal{L}^n) = \dim_E \Gamma(X(\mu), \mathcal{L}^n).$$

Apparently, this conjecture would not be very useful unless one can say something about the line bundle \mathcal{L} . In fact, the conjecture in [PR3] is different and more precise. Namely, in the *loc. cit.*, they constructed some line bundle \mathcal{L}_1 on the affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ and some line bundle \mathcal{L}_2 on $X(\mu)$, both of which are explicit and are purely in terms of group theory (see §2.3 for the precise construction). Then they conjectured

The Coherence Conjecture. *For $n \gg 0$,*

$$\dim_{\bar{k}} \Gamma(\mathcal{A}(\mu), \mathcal{L}_1^n) = \dim_E \Gamma(X(\mu), \mathcal{L}_2^n).$$

In addition, in *loc. cit.*, for certain groups, they constructed natural ample line bundles \mathcal{L} on the corresponding local models, whose restrictions give \mathcal{L}_1 and \mathcal{L}_2 .

What is good for the coherence conjecture? First of all, the conjecture is group theoretical, i.e. the statement is uniform for all groups. The non-group theoretical part then is absorbed into the construction of natural line bundles on local models and the identification of their restrictions with the group theoretically constructed line bundles. This is a much simpler problem. An example is illustrated in §8. More importantly, the right hand side in the coherence conjecture is defined over \mathcal{O}_E and therefore, it is equivalent to prove that

$$\dim_{\bar{k}} \Gamma(\mathcal{A}(\mu), \mathcal{L}_1^n) = \dim_{\bar{k}} \Gamma(X(\mu), \mathcal{L}_2^n).$$

Observe that in the above formulation, everything is over the field k rather than over a mixed characteristic ring. That is, we are dealing with algebraic geometry rather than arithmetic!

How can we prove this conjecture? Suppose that we can find a scheme $\overline{\text{Gr}}_{\mathcal{G},\mu}$ (the reason we choose this notation will be clear soon), which is flat over $\bar{k}[t]$, together with a line bundle \mathcal{L} such that its fiber over $0 \in \mathbb{A}^1$ is $(\mathcal{A}(\mu), \mathcal{L}_1)$ and its fiber over $y \neq 0$ is $(X(\mu), \mathcal{L}_2)$, then the coherence conjecture will follow. In fact, such $\overline{\text{Gr}}_{\mathcal{G},\mu}$ does exist and can be constructed purely group theoretically. They are the (generalized) equal characteristic counterparts of local models, which we will call the global Schubert varieties. Let us briefly indicate the construction of $\overline{\text{Gr}}_{\mathcal{G},\mu}$ here (the construction of the line bundle \mathcal{L} , which we ignore here, is also purely group theoretical, see §4). For simplicity, let us assume that G' is split over k (the non-split case will also be considered in the paper). Let B be a Borel subgroup of G' . Then in [G], Gaitsgory (following the idea of Kottwitz and Beilinson) constructed a family

of ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ over \mathbb{A}^1 , which is a deformation from the affine Grassmannian $\mathrm{Gr}_{G'}$ of G' to the affine flag variety $\mathcal{F}\ell_{G'}$ of G' . By its construction,

$$\mathrm{Gr}_{\mathcal{G}}|_{\mathbb{G}_m} \cong (\mathrm{Gr}_{G'} \times G'/B) \times \mathbb{G}_m^1, \quad \mathrm{Gr}_{\mathcal{G}}|_0 \cong \mathcal{F}\ell_{G'}.$$

When μ is minuscule, the Schubert variety $\overline{\mathrm{Gr}}_{\mu}$ corresponds to μ in $\mathrm{Gr}_{G'}$ is in fact isomorphic to $X(\mu)$. In addition, we can "spread it out" over \mathbb{G}_m as $(\overline{\mathrm{Gr}}_{\mu} \times *) \times \mathbb{G}_m$ to get a closed subscheme of $\mathrm{Gr}_{\mathcal{G}}|_{\mathbb{G}_m}$, where $*$ is the base point in G'/B . Now define $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ as the closure of $(\overline{\mathrm{Gr}}_{\mu} \times *) \times \mathbb{G}_m$ inside $\mathrm{Gr}_{\mathcal{G}}$. By definition, its fiber over $y \neq 0$ is isomorphic to $X(\mu)$. On the other hand, it is not hard to see that $\mathcal{A}(\mu) \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0$ (cf. Lemma 3.8). Therefore, the coherence conjecture will follow if we can show that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0 = \mathcal{A}(\mu)$ (and if we can construct the corresponding line bundle).

At the first sight, it seems the idea is circular. However, it is not the case. The reason, as we mentioned before, is that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ now is a scheme over k and we have much more tools to attack the problem. Observe that to prove that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0 = \mathcal{A}(\mu)$, we need to show that

- (1) $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0)_{\mathrm{red}} = \mathcal{A}(\mu)$ (Theorem 3.9);
- (2) $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0$ is reduced (Theorem 3.10).

Part (1) can be achieved by the calculation of the nearby cycle $\mathcal{Z}_{\mu} = \Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathbb{Q}_{\ell})$ of the family $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ (see Lemma 7.1). Usually, this is a hard problem. The miracle here is that if \mathcal{Z}_{μ} is regard as an object in the category of Iwahori equivariant perverse sheaves on $\mathcal{F}\ell_{G'}$, it has very nice properties. Namely, by the main result of [G] (in the case when G' is split), \mathcal{Z}_{μ} is a central sheaf, i.e. for any other Iwahori equivariant perverse sheaf \mathcal{F} on $\mathcal{F}\ell_{G'}$, the convolution product $\mathcal{Z}_{\mu} \star \mathcal{F}$ (see (7.1.1)-(7.1.2) for the definition) is perverse and

$$\mathcal{Z}_{\mu} \star \mathcal{F} \cong \mathcal{F} \star \mathcal{Z}_{\mu}.$$

Then by a result of Arkhipov-Bezrukavnikov [AB, Theorem 4], the above properties put a strong restrictions of the support of \mathcal{Z}_{μ} , which will imply Part (1). We shall mention that although we assume here that G' is split, the same strategy can be applied to the non-split groups. This is done in §7, where we generalize the results of [G] and [AB] to ramified groups as well. Our arguments are simpler than the originally ones in [G, AB], and will have the following technical advantage. As we mentioned above, $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ should be regarded as the equal characteristic counterparts of local models. Therefore, it is natural (and indeed important) to determine the nearby cycles $\Psi_{\mathcal{M}_{K,\{\mu\}}^{\mathrm{loc}}}(\mathbb{Q}_{\ell})$ for the local models. For example, if one could prove that these sheaves are also central (the *Kottwitz conjecture*²), then one could conclude (1.0.1) directly. It turns out the arguments in §7 have a direct generalization to the mixed characteristic situation and in a joint work with Pappas [PZ], we use it to solve the Kottwitz conjecture (some previous cases are proved by Haines and Ngô [HN]).

Now we turn to Part (2), which is more difficult. The idea is that we can assume $\mathrm{char} k > 0$ and use the powerful technique of Frobenius splitting (cf. [MR]). To prove that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0$ is reduced, it is enough to prove that it is Frobenius split. To achieve this goal, we embeds $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ into some larger scheme $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\lambda}^{BD}$ over \mathbb{A}^1 , which

¹In the main body of this paper, we will work with a different family so that this extra G'/B factor does not appear.

²In fact, the Kottwitz conjecture is weaker than this statement, and its significance lies in the Langlands-Kottwitz method to calculate the Zeta functions of Shimura varieties.

is a closed subscheme of a version of the Beilinson-Drinfeld Grassmannian. The scheme $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{BD}$ is normal and its fiber over 0 is reduced. Then to prove that

$$\overline{\text{Gr}}_{\mathcal{G},\mu}|_0 = \overline{\text{Gr}}_{\mathcal{G},\mu} \cap \overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{BD}|_0$$

is Frobenius split, it is enough to construct a Frobenius splitting of $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{BD}$, compatible with $\overline{\text{Gr}}_{\mathcal{G},\mu}$ and $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{BD}|_0$. Since $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{BD}$ is normal, we can prove this just for some nice open subscheme $U \subset \overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{BD}$, such that $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{BD} - U$ has codimension two. In particular, the open subscheme U will not intersect with $\overline{\text{Gr}}_{\mathcal{G},\mu}|_0$, which is of our primary interest! Section 6 is devoted to realizing this idea.

Now let us describe the organization of the paper and some other results proved in the paper.

In §2, we review the coherence conjecture of Pappas and Rapoport. In §2.1, we review some of basic theories of reductive groups over local field and introduce various notations used in the rest of the paper. In §2.2, we rapidly recall the main results of [PR3] (and [Fa]) concerning the loop groups and the geometry of their flag varieties. In §2.3, we state the main theorems (Theorem 1 and 2) of our paper, which are the modified version of original coherence conjecture of Pappas and Rapoport (see Remark 2.1 for the reason of the modification).

In §3, we introduce the main geometrical object we are going to study in the paper, namely, the global Schubert varieties. They are varieties projective over the affine line \mathbb{A}^1 , which are the counterparts of local models in the equal characteristic situation. In §3.1, we define the global affine Grassmannian over a curve for general (non-constant) group schemes. After the work of [PR3, PR5, He], this construction is now standard. In §3.2, we construct a special Bruhat-Tits group scheme over \mathbb{A}^1 , i.e. a group scheme which is only ramified at the origin. Let us remark similar constructions are also considered in [HNY, Ri]. In §3.3, we apply construction of the global affine Grassmannian to the group scheme we consider in the paper. We introduce the global Schubert variety $\overline{\text{Gr}}_{\mathcal{G},\mu}$ associate to any geometrical conjugacy class of 1-parameter subgroup μ of G . We state another main theorem (Theorem 3) which asserts that the special fiber of $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is $\mathcal{A}^Y(\mu)$, and show that the variety $\mathcal{A}^Y(\mu)$ is contained in it (Lemma 3.8). In §4, we explain that our assertion about the special fiber of $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is equivalent to the coherence conjecture. The key ingredient is a certain line bundle on the global affine Grassmannian, namely, the pullback of the determinant line bundle along the closed embedding

$$\text{Gr}_{\mathcal{G}} \rightarrow \text{Gr}_{\text{Lie}(\mathcal{G})}.$$

We calculate its central charges at each fiber (which turn out to be twice of the dual Coxeter number) and find the remarkable fact that the central charge of line bundles on the global affine Grassmannians are constant along the curve (Proposition 4.1).

In §5, we make some preparations towards the proof of our main theorem. We study two basic geometrical structures of $\overline{\text{Gr}}_{\mathcal{G},\mu}$: (i) in §5.2, we will construct certain affine charts of $\overline{\text{Gr}}_{\mathcal{G},\mu}$, which turn out to be isomorphic to affine spaces over \tilde{C} ; and (ii) in §5.3, we will construct a \mathbb{G}_m -action on $\overline{\text{Gr}}_{\mathcal{G},\mu}$, so that the map $\overline{\text{Gr}}_{\mathcal{G},\mu} \rightarrow \tilde{C}$ is \mathbb{G}_m -equivariant, where \mathbb{G}_m acts on $\tilde{C} = \mathbb{A}^1$ by natural dilatation. To establish (i), we will need to first construct the global root subgroups of $\mathcal{L}\mathcal{G}$ as in §5.1.

The next two sections are then devoted to the proof of the theorem concerning the special fiber of $\overline{\text{Gr}}_{\mathcal{G},\mu}$, as been already outlined as above. The first part of the proof, presented in §6, concerns the scheme theoretical structure of the special

fiber. Namely, we prove that it is reduced. This is achieved by the technique of Frobenius splitting. As a warm up, we prove in §6.1 that Theorem 1 is a special case of Theorem 2, which should be well-known to experts. Then we introduce the Beilinson-Drinfeld Grassmannian and the convolution Grassmannian and reduce Theorem 3.10 to Theorem 6.10. In §6.3, we prove a special case of Theorem 6.10 by studying the affine flag variety associated to a special parahoric group scheme. Recall that a result of Beilinson-Drinfeld (cf. [BD, §4.6]) asserts that the Schubert varieties in the affine Grassmannian are Gorenstein. We examine in §6.3 that to what extent this result holds for ramified groups (i.e. reductive groups split over a ramified extension). It turns out this result extends to all affine flag varieties associated to special parahorics except the case the special parahoric is a parahoric of the ramified odd unitary group SU_{2n+1} , whose special fiber has reductive quotient SO_{2n+1} (Theorem 6.13). In this exceptional case, no Schubert variety of positive dimension in the corresponding affine flag variety is Gorenstein (Remark 6.2). In §7, we give the second part of the proof, which asserts that topologically, the special fiber of $\overline{\text{Gr}}_{\mathcal{G},\mu}$ coincides with $\mathcal{A}^Y(\mu)$. This is achieved by the description of the support of the nearby cycle (for the intersection cohomology sheaf) of this family. In the case when the group is split, this follows the earlier works of [G] and [AB]. In §7.1, we generalize their results to ramified groups, with certain simplifications of the original arguments. In §7.2, we study the monodromy of this nearby cycle. This is not needed for the main results of the current paper. But it is expected to have important applications elsewhere.

The paper has two appendices. The first one, §8, calculates the line bundles on the local models for the ramified unitary groups. The study of these local models are the main motivation for Pappas and Rapoport to make the coherence conjecture. Since their original conjecture is not as stated in our main theorem, we explained in this appendix why our main theorem is correct for the applications to local models. The second appendix (§9) contains certain algebro-geometrical constructions used in the main body of the paper.

Notations. Let k be a field, and fix \bar{k} be an algebraic closure of k . We will denote $k^s \subset \bar{k}$ the separable closure of k in \bar{k} .

Let X be a Y -scheme and $V \rightarrow Y$ be a morphism, the base change $X \times_Y V$ is denoted by X_V or $X|_V$. If $V = \text{Spec} R$, it is sometimes also denoted by X_R . If $V = x = \text{Spec} k$ is a point, then it is sometimes also denoted by $(X)_x$.

Let \mathcal{V} be a vector bundle on a scheme V , then we denote $\det(\mathcal{V})$ to be the top exterior wedge of \mathcal{V} , which is a line bundle.

If A is an affine algebraic group (not necessarily a torus) over a field k , we denote $\mathbb{X}^\bullet(A)$ (resp. $\mathbb{X}_\bullet(A)$) to be its character group (resp. cocharacter group) over k^s . The Galois group $\Gamma = \text{Gal}(k^s/k)$ acts on $\mathbb{X}^\bullet(A)$ (resp. $\mathbb{X}_\bullet(A)$) and the invariants (resp. coinvariants) are denoted by $\mathbb{X}^\bullet(A)^\Gamma$ (resp. $\mathbb{X}^\bullet(A)_\Gamma, \mathbb{X}_\bullet(A)^\Gamma, \mathbb{X}_\bullet(A)_\Gamma$).

Let \mathcal{G} be a flat group scheme over V , the trivial \mathcal{G} -torsor (i.e. \mathcal{G} itself regarded as a \mathcal{G} -torsor by right multiplication) is denoted by \mathcal{E}^0 . For a \mathcal{G} -torsor \mathcal{E} , we use $\text{ad } \mathcal{E}$ to denote the associated adjoint bundle. If \mathcal{P} is a \mathcal{G} torsor and X is a scheme over V with an action of \mathcal{G} , we denote the twisted product by $\mathcal{P} \times^{\mathcal{G}} X$, which is the quotient of $\mathcal{P} \times_V X$ by the diagonal action \mathcal{G} .

Let G be a reductive group over a field. We denote G_{der} to be its derived group, G_{sc} to be the simply-connected cover of G_{der} and G_{ad} to be the adjoint group.

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2. REVIEW OF THE LOCAL PICTURE, FORMULATION OF THE CONJECTURE

2.1. Group theoretical data. Let k be an algebraically closed field. Let $\mathcal{O} = k[[t]]$ and $F = k((t))$. Let $\Gamma = \text{Gal}(F^s/F)$ be the inertial (Galois) group, where F^s is the separable closure of F . Let us emphasize that we choose a uniformizer t . Let G be a connected reductive group over F . In this paper, unless otherwise stated, we will assume that G splits over a *tamely* ramified extension \tilde{F}/F .

Let S be a maximal F -split torus of G . Let $T = \mathcal{Z}_G(S)$ be the centralizer of S in G , which is a maximal torus of G since G is quasi-split over F . Let us choose a rational Borel subgroup $B \supset T$. Let H be a split Chevalley group over \mathbb{Z} such that $H \otimes F^s \cong G \otimes F^s$. We need to choose this isomorphism carefully. Let us fix a pinning (H, B_H, T_H, X) of H over \mathbb{Z} . Let us recall that this means that B_H is a Borel subgroup of H , T_H is a split maximal torus contained in B_H , and $X = \sum_{\tilde{a} \in \Delta} X_{\tilde{a}} \in \text{Lie} B$, where Δ is the set of simple roots, $\tilde{U}_{\tilde{a}}$ is the root subgroup corresponding to \tilde{a} and $X_{\tilde{a}}$ is a generator in the rank one free \mathbb{Z} -module $\text{Lie} \tilde{U}_{\tilde{a}}$. Let us choose an isomorphism $(G, B, T) \otimes_F \tilde{F} \cong (H, B_H, T_H) \otimes_{\mathbb{Z}} \tilde{F}$, where \tilde{F}/F is a cyclic extension such that $G \otimes \tilde{F}$ splits. This induces an isomorphism of the root data $(\mathbb{X}^\bullet(T_H), \Delta, \mathbb{X}_\bullet(T_H), \Delta^\vee) \cong (\mathbb{X}^\bullet(T), \Delta, \mathbb{X}_\bullet(T), \Delta^\vee)$. Let Ξ be the group of pinned automorphisms of (H, B_H, T_H, X) , which is canonically isomorphic to the group of the automorphisms of the root datum $(\mathbb{X}^\bullet(T_H), \Delta, \mathbb{X}_\bullet(T_H), \Delta^\vee)$.

Now the action of $I = \text{Gal}(\tilde{F}/F)$ on $G \otimes_F \tilde{F}$ induces a homomorphism $\psi : I \rightarrow \Xi$. Then we can always choose an isomorphism

$$(2.1.1) \quad (G, B, T) \otimes_F \tilde{F} \cong (H, B_H, T_H) \otimes_{\mathbb{Z}} \tilde{F}$$

such that the action of $\gamma \in I$ on the left hand side corresponds to $\psi(\gamma) \otimes \gamma$. In the rest of the paper, we fix such an isomorphism. This determines a point v_0 in $A(G, S)$, the apartment associated to (G, S) ([BT1])³. This is a special point of $A(G, S)$, which in turn gives a parahoric group scheme \mathcal{G}_{v_0} over \mathcal{O} , namely

$$(2.1.2) \quad \mathcal{G}_{v_0} := ((\text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}}(H \otimes \mathcal{O}_{\tilde{F}}))^{\Gamma})^0.$$

Let us explain the notations. Here Res stands for the Weil restriction, so that $\text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}}(H \otimes \mathcal{O}_{\tilde{F}})$ is a smooth group scheme over \mathcal{O} (cf. [Ed, 2.2]), with an action of Γ . The notation $(-)^{\Gamma}$ stands for taking the Γ -fixed point subscheme. Under our tameness assumption, $\tilde{\mathcal{G}}_{v_0} := (\text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}}(H \otimes \mathcal{O}_{\tilde{F}}))^{\Gamma}$ is smooth by [Ed, 3.4]. Finally, $(-)^0$ stands for taking the neutral connected component. I.e., \mathcal{G}_{v_0} and $\tilde{\mathcal{G}}_{v_0}$ have the same generic fiber and the special fiber of \mathcal{G}_{v_0} is the neutral connected component of the special fiber of $\tilde{\mathcal{G}}_{v_0}$.

Recall that $A(G, S)$ is an affine space under $\mathbb{X}_\bullet(S)_{\mathbb{R}}$. For every facet $\sigma \subset A(G, S)$, let \mathcal{G}_σ be the parahoric group scheme over \mathcal{O} (in particular, the special fiber of \mathcal{G}_σ is connected). Let C be the alcove in $A(G, S)$, whose closure contains the point v_0 , and is contained in the finite Weyl chamber determined by the chosen Borel. This determines a set of simple affine roots $\alpha_i, i \in \mathbf{S}$, where \mathbf{S} is the set of vertices of the affine Dynkin diagram associated to G .

³More precisely, v_0 is a point in the apartment associated to the adjoint group $(G_{\text{ad}}, S_{\text{ad}})$. But since in the paper, we only use the combinatoric structures of $A(G, S)$, we will not distinguish it from the one associated to the adjoint group.

Let \widetilde{W} be the Iwahori-Weyl group of G (cf. [HR]), which acts on $A(G, S)$. This is defined to be $\mathcal{N}_G(S)(F)/\ker \kappa$, where

$$(2.1.3) \quad \kappa : T(F) \rightarrow \mathbb{X}_\bullet(T)_\Gamma$$

is the Kottwitz homomorphism (cf. [Ko, §7]). One has the following

$$(2.1.4) \quad 1 \rightarrow \mathbb{X}_\bullet(T)_\Gamma \rightarrow \widetilde{W} \rightarrow W_0 \rightarrow 1,$$

where W_0 is the relative Weyl group of G over F . In what follows, we use t_λ to denote the translation element in \widetilde{W} given by $\lambda \in \mathbb{X}_\bullet(T)_\Gamma$ from the above map (2.1.4)⁴. But occasionally, we also use λ itself to denote this translation element if no confusion is likely to arise. The pinned isomorphism (2.1.1) determines a set of positive roots $\Phi^+ = \Phi(G, S)^+$ for G . There is a natural map $\mathbb{X}_\bullet(T)_\Gamma \rightarrow \mathbb{X}_\bullet(S)_\mathbb{R}$. We define

$$(2.1.5) \quad \mathbb{X}_\bullet(T)_\Gamma^+ = \{\lambda \mid (\lambda, a) \geq 0 \text{ for } a \in \Phi^+\}.$$

A choice of a special vertex (e.g. the point v_0) of $A(G, S)$ gives a splitting of the exact sequence and, therefore we can write $w = t_\lambda w_f$ for $\lambda \in \mathbb{X}_\bullet(T)_\Gamma$ and $w_f \in W_0$.

Let W_{aff} be the affine Weyl group of G , i.e. the Iwahori-Weyl group of G_{sc} , which is a Coxeter group. One has

$$1 \rightarrow \mathbb{X}_\bullet(T_{\text{sc}})_\Gamma \rightarrow W_{\text{aff}} \rightarrow W_0 \rightarrow 1,$$

where T_{sc} is the inverse image of T in G_{sc} . One can write $\widetilde{W} = W_{\text{aff}} \rtimes \Omega$, where Ω is the subgroup of \widetilde{W} that fixes the chosen alcove C . This gives \widetilde{W} a quasi Coxeter group structure, and it makes sense to talk about the length of an element $w \in \widetilde{W}$ and there is a Bruhat order on \widetilde{W} . Namely, if we write $w_1 = w'_1 \tau_1, w_2 = w'_2 \tau_2$ with $w'_i \in W_{\text{aff}}, \tau_i \in \Omega$, then $\ell(w_i) = \ell(w'_i)$ and $w_1 \leq w_2$ if and only if $\tau_1 = \tau_2$ and $w'_1 \leq w'_2$. A lot of combinatorics of the Iwahori-Weyl group arises from the study of the restriction of the length function and the Bruhat order to $\mathbb{X}_\bullet(T)_\Gamma \subset \widetilde{W}$. Some of them will be reviewed in Lemma 2.1-2.2.

Now let us recall the definition of the *admissible set* in the Iwahori-Weyl group. Let \bar{W} be the absolute Weyl group of G , i.e. the Weyl group for (H, T_H) . Let $\mu : (\mathbb{G}_m)_{\bar{F}} \rightarrow G \otimes \bar{F}$ be a geometrical conjugacy class of 1-parameter subgroup. It determines a \bar{W} -orbit in $\mathbb{X}_\bullet(T)$. One can associate μ a W_0 -orbits Λ in $\mathbb{X}_\bullet(T)_\Gamma$ as follows. Choose a Borel subgroup of G containing T , and defined over F . This gives a unique element in this \bar{W} -orbit, still denoted by μ , which is dominant w.r.t. this Borel subgroup. Let $\bar{\mu}$ be its image in $\mathbb{X}_\bullet(T)_\Gamma$, and let $\Lambda = W_0 \bar{\mu}$. It turns out Λ does not depend on the choice of the rational Borel subgroup of G , since any two such F -rational Borels that contain T will be conjugate to each other by an element in W_0 . For $\mu \in \mathbb{X}_\bullet(T)$, define the admissible set

$$(2.1.6) \quad \text{Adm}(\mu) = \{w \in \widetilde{W} \mid w \leq t_\lambda, \text{ for some } \lambda \in \Lambda\}.$$

Under the map $\mathbb{X}_\bullet(T)_\Gamma \rightarrow \widetilde{W} \rightarrow \widetilde{W}/W_{\text{aff}} \cong \Omega$, the set Λ maps to a single element (cf. [R2, Lemma 3.1]), denoted by τ_μ . Define

$$\text{Adm}(\mu)^\circ = \tau_\mu^{-1} \text{Adm}(\mu).$$

For $Y \subset \mathbf{S}$ any subset. Let W^Y denote the subgroup of W_{aff} generated by $\{r_i, i \in \mathbf{S} - Y\}$, where r_i is the simple reflection corresponding to i . Then set

$$\text{Adm}^Y(\mu) = W^Y \text{Adm}(\mu) W^Y \subset \widetilde{W},$$

⁴Note that under the sign convention of the Kottwitz homomorphism in [Ko], t_λ acts on $\mathbb{X}_\bullet(S)_\mathbb{R}$ by $v \mapsto v - \lambda$.

and

$$\mathrm{Adm}^Y(\mu)^\circ = \tau_\mu^{-1} \mathrm{Adm}^Y(\mu).$$

Note that $\mathrm{Adm}^Y(\mu)^\circ \subset W_{\mathrm{aff}}$.

Let G_{ad} be the adjoint group of G . Then the above construction applies to G_{ad} . Therefore, for each $\mu \in \mathbb{X}_\bullet(T_{\mathrm{ad}})$ (where T_{ad} is the image of T in G_{ad}), one can associate a W_0 -orbit $\Lambda_{\mathrm{ad}} \in \mathbb{X}_\bullet(T_{\mathrm{ad}})_\Gamma$ and $\mathrm{Adm}^Y(\mu) \in \widetilde{W}_{\mathrm{ad}}$.

We recall a few combinatorics about the Iwahori-Weyl group which will be used in the sequel. They are well-known in the case G is split, and are extended to ramified groups by Richarz [Ri].

Let 2ρ be the sum of all positive roots (for H). Observe that given $\mu \in \mathbb{X}_\bullet(T)_\Gamma$, we have $(\mu, \sum_{a \in \Phi^+} a) = (\tilde{\mu}, 2\rho)$ for any of its lift $\tilde{\mu} \in \mathbb{X}_\bullet(T)$. By abuse of notation, we denote this number by $(\mu, 2\rho)$.

Lemma 2.1. *Let $\mu \in \mathbb{X}_\bullet(T)_\Gamma^+$.*

- (1) *Let $\Lambda \subset \mathbb{X}_\bullet(T)_\Gamma$ be the subset associated to μ as above. Then for all $\nu \in \Lambda$,*
 $\ell(t_\nu) = (2\rho, \mu)$.
- (2) $\ell(t_\mu w_f) = \ell(t_\mu) + \ell(w_f)$.

Proof. See [Ri] formula (1.9) and Corollary 1.8. Note that the same statement of (2) in [Ri] holds for anti-dominant element rather than the dominant element as stated here. The reason is due to the different normalization of the Kottwitz homomorphism. See Footnote 4. \square

On the finitely generated abelian group $\mathbb{X}_\bullet(T)_\Gamma$, there are two partial orders. One is the restriction of the Bruhat order on \widetilde{W} , denoted as " \leq ". The other, denoted by " \preceq ", is defined as follows. Recall that the lattice $\mathbb{X}_\bullet(T_{\mathrm{sc}})$ is the coroot lattice for H . This Galois group Γ acts on $\mathbb{X}_\bullet(T_{\mathrm{sc}})$ which sends the positive coroots of H (determined by the pinning) to positive coroots. Therefore, it makes sense to talk about positive elements in $\mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$. Namely, $\lambda \in \mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$ is positive if its preimage in $\mathbb{X}_\bullet(T_{\mathrm{sc}})$ is a sum of positive coroots (of (H, T_H)). Since $\mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma \subset \mathbb{X}_\bullet(T)_\Gamma$, we can define $\lambda \preceq \mu$ if $\mu - \lambda$ is positive in $\mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$. On the other hand, there is the Bruhat order in \widetilde{W} defined by the alcove C . We have the following proposition.

Lemma 2.2. *Let $\lambda, \mu \in \mathbb{X}_\bullet(T)_\Gamma^+$. Then $\lambda \preceq \mu$ if and only if $t_\lambda \leq t_\mu$ in the Bruhat order.*

Proof. In the case that G is split, the proof is contained in [R2, Proposition 3.2, 3.5]. The ramified case can be reduced to the same proof as shown in [Ri, Corollary 1.8]. \square

2.2. Loop groups and their flag varieties. Let $\sigma \subset A(G, S)$ be a facet. Let

$$\mathcal{F}\ell_\sigma = LG/L^+\mathcal{G}_\sigma$$

be the (partial) flag variety of LG . Let us recall that LG is the loop group of G , which represents the functor which associates to every k -algebra R the group $G(R((t)))$, $L^+\mathcal{G}_\sigma$ is the path group of \mathcal{G}_σ , which represents the functor which associates to every k -algebra R the group $\mathcal{G}_\sigma(R[[t]])$, and $\mathcal{F}\ell_\sigma = LG/L^+\mathcal{G}_\sigma$ is the *fppf* quotient. Let us also recall that LG is represented by an ind-affine scheme, $L^+\mathcal{G}_\sigma$ is represented by an affine scheme, which is a closed subscheme of LG , and $\mathcal{F}\ell_\sigma$ is represented by an ind-scheme, ind-projective over k . Denote $B = L^+\mathcal{G}_C$ the Iwahori subgroup of LG , and denote $\mathcal{F}\ell_C$ by $\mathcal{F}\ell$, which we call the affine flag variety of G . If G splits over F , so that $G = H \otimes F$, then the special vertex v_0 is hyperspecial, and corresponds to the parahoric group scheme $H \otimes k[[t]]$. Then we denote $\mathcal{F}\ell_{v_0}$ by Gr_H and call it the

affine Grassmannian of H . Let $Y \subset \mathbf{S}$ be a subset, and $\sigma_Y \subset A(G, S)$ be the facet such that $\alpha_i(\sigma_Y) = 0$ for $i \in \mathbf{S} - Y$. Observe that $\sigma_{\mathbf{S}} = C$ is the chosen alcove. We also denote $\mathcal{F}\ell_{\sigma_Y}$ by $\mathcal{F}\ell^Y$ for simplicity.

Let us recall that B -orbits of $\mathcal{F}\ell$ are parameterized by \widetilde{W} . In general, the $L^+\mathcal{G}_{\sigma_Y}$ -orbits of $\mathcal{F}\ell^{Y'}$ are parameterized by $W^Y \setminus \widetilde{W}/W^{Y'}$, where W^Y is the Weyl group of $\mathcal{G}_{\sigma_Y} \otimes k$. For $w \in \widetilde{W}$, let ${}^Y\mathcal{F}\ell_w^{Y'} \subset \mathcal{F}\ell^{Y'}$ denote the corresponding Schubert variety, i.e. the closure of the $L^+\mathcal{G}_{\sigma_Y}$ -orbit through w . If $Y = Y'$, then we simply denote it by $\mathcal{F}\ell_w^Y$. If G is split, and $\mathcal{G} = H \otimes k[[t]]$ is a hyperspecial model, recall that $L^+\mathcal{G}$ -orbits of Gr_H are parameterized by $\widetilde{W} \setminus \widetilde{W}/\widetilde{W} \cong \mathbb{X}_{\bullet}(T)^+$, the set of dominant coweights of G . For $\mu \in \mathbb{X}_{\bullet}(T)^+$, let $\overline{\mathrm{Gr}}_{\mu}$ be the corresponding Schubert variety in Gr_H .

Let us recall the following result of [Fa, PR3].

Theorem 2.3. *Let $p = \mathrm{char} k$. Assume that $p \nmid |\pi_1(G_{\mathrm{der}})|$, where G_{der} is the derived group of G . Then the Schubert variety $\mathcal{F}\ell_w^Y$ is normal, has rational singularities, and is Frobenius-split if $p > 0$.*

For $\mu \in \mathbb{X}_{\bullet}(T)$, let

$$\mathcal{A}^Y(\mu)^{\circ} = \bigcup_{w \in \mathrm{Adm}^Y(\mu)^{\circ}} {}^Y\mathcal{F}\ell_{\mathrm{sc}, w}^Y,$$

where $\sigma_{Y^{\circ}} = \tau_{\mu}^{-1}(\sigma_Y)$, and where ${}^Y\mathcal{F}\ell_{\mathrm{sc}, w}^Y$ is the union of Schubert varieties (more precisely, the closure of $L^+\mathcal{G}_{\sigma_{Y^{\circ}}}$ -orbits) in the partial affine flag variety $\mathcal{F}\ell_{\mathrm{sc}}^Y = LG_{\mathrm{sc}}/L^+\mathcal{G}_{\sigma_Y}$. Then $\mathcal{A}^Y(\mu)^{\circ}$ is a reducible subvariety of $\mathcal{F}\ell_{\mathrm{sc}}^Y$, with irreducible components

$${}^Y\mathcal{F}\ell_{\mathrm{sc}, \tau_{\mu}^{-1}t_{\lambda}}^Y, \quad \lambda \in \Lambda \subset \mathbb{X}_{\bullet}(T)_{\Gamma} \subset \widetilde{W}.$$

When $p \nmid |\pi_1(G_{\mathrm{der}})|$, it is also convenient to consider

$$\mathcal{A}^Y(\mu) = \bigcup_{w \in \mathrm{Adm}^Y(\mu)} \mathcal{F}\ell_w^Y.$$

Choosing a lift $g \in G(F)$ of $\tau_{\mu} \in \widetilde{W}$ and identifying $\mathcal{F}\ell_{\mathrm{sc}}^Y$ with the reduced part of the neutral connected component of $\mathcal{F}\ell^Y$ (see [PR3, §6]), we can define a map $\mathcal{F}\ell_{\mathrm{sc}}^Y \rightarrow \mathcal{F}\ell^Y$, $x \mapsto gx$. Clearly, this map induces an isomorphism $\mathcal{A}^Y(\mu)^{\circ} \cong \mathcal{A}^Y(\mu)$.

In particular, if $G = H \otimes F$ is split and $\sigma_Y = v_0$ is the hyperspecial vertex corresponding to $H \otimes \mathcal{O}$, then $\mathcal{A}^Y(\mu)^{\circ}$ is denoted by $\mathrm{Gr}_{\leq \mu}$, so that if $p \nmid |\pi_1(G_{\mathrm{der}})|$, then we have the isomorphism $\mathrm{Gr}_{\leq \mu} \cong \overline{\mathrm{Gr}}_{\mu}$.

We also need to review the Picard group of $\mathcal{F}\ell$. For simplicity, we assume that G is simple, simply-connected, absolutely simple. In this case $\mathcal{F}\ell$ is connected. For each $i \in \mathbf{S}$, let P_i be the corresponding parahoric subgroup containing B so that $P_i/B \cong \mathbb{P}^1$. This \mathbb{P}^1 maps naturally to $\mathcal{F}\ell$ via $P_i \rightarrow LG$, and the image will be denoted as \mathbb{P}_i^1 . Then it is known ([PR3, §10]) that there is a unique line bundle $\mathcal{L}(\epsilon_i)$ on $\mathcal{F}\ell$, whose restriction to the \mathbb{P}_i^1 is $\mathcal{O}_{\mathbb{P}^1}(1)$, and whose restrictions to other \mathbb{P}_j^1 s with $j \neq i$ is trivial. Then there is an isomorphism

$$\mathrm{Pic}(\mathcal{F}\ell) \cong \bigoplus_{i \in \mathbf{S}} \mathbb{Z}\mathcal{L}(\epsilon_i).$$

Let us write $\otimes_i \mathcal{L}(\epsilon_i)^{n_i}$ as $\mathcal{L}(\sum_i n_i \epsilon_i)$. As explained in *loc. cit.*, ϵ_i can be thought as the fundamental weights of the Kac-Moody group associated to LG , and therefore, $\mathrm{Pic}(\mathcal{F}\ell)$ is identified with the weight lattice of the corresponding Kac-Moody group.

There is also a morphism

$$(2.2.1) \quad c : \text{Pic}(\mathcal{F}\ell) \rightarrow \mathbb{Z}$$

called the central charge. If we identify $\mathcal{L} \in \text{Pic}(\mathcal{F}\ell)$ with a weight of the corresponding Kac-Moody group, then $c(\mathcal{L})$ is just the restriction of this weight to the central \mathbb{G}_m in the Kac-Moody group. Explicitly,

$$(2.2.2) \quad c(\mathcal{L}(\epsilon_i)) = a_i^\vee,$$

where $a_i^\vee (i \in \mathbf{S})$ are defined as in [Kac, §6.1]. The kernel of c can be described as follows. Let s denote the closed point of $\text{Spec}\mathcal{O}$, and let $(\mathcal{G}_C)_s$ denote the special fiber of \mathcal{G}_C . Recall that for any k -algebra R , $\mathcal{F}\ell(R)$ represents the set of \mathcal{G}_C torsors on $\text{Spec}R[[t]]$ together with a trivialization over $\text{Spec}R((t))$. Therefore, by restriction of the \mathcal{G}_C -torsors over $\text{Spec}R \subset \text{Spec}R[[t]]$, we obtain a natural morphism $\mathcal{F}\ell \rightarrow \mathbb{B}(\mathcal{G}_C)_s$ (here $\mathbb{B}(\mathcal{G}_C)_s$ is the classifying stack of $(\mathcal{G}_C)_s$), which induces $\mathbb{X}^\bullet((\mathcal{G}_C)_s) \cong \text{Pic}(\mathbb{B}(\mathcal{G}_C)_s) \rightarrow \text{Pic}(\mathcal{F}\ell)$. We have the short exact sequence

$$(2.2.3) \quad 0 \rightarrow \mathbb{X}^\bullet((\mathcal{G}_C)_s) \rightarrow \text{Pic}(\mathcal{F}\ell) \xrightarrow{c} \mathbb{Z} \rightarrow 0.$$

Now let $Y \subset \mathbf{S}$ be a non-empty subset. Observe that if $\mathcal{L}(\sum n_i \epsilon_i)$ is a line bundle on $\mathcal{F}\ell$, with $n_i = 0$ for $i \in \mathbf{S} - Y$, then this line bundle is the pullback of a unique line bundle along $\mathcal{F}\ell \rightarrow \mathcal{F}\ell^Y$, denoted by $\mathcal{L}^Y(\sum_{i \in Y} n_i \epsilon_i)$. In this way, we have

$$\text{Pic}(\mathcal{F}\ell^Y) \cong \bigoplus_{i \in Y} \mathbb{Z} \mathcal{L} \epsilon_i.$$

The central charge of a line bundle \mathcal{L} on $\mathcal{F}\ell^Y$ is defined to be the central charge of its pullback to $\mathcal{F}\ell$, i.e. the image of \mathcal{L} under $\text{Pic}(\mathcal{F}\ell^Y) \rightarrow \text{Pic}(\mathcal{F}\ell) \xrightarrow{c} \mathbb{Z}$. Observe that $\mathcal{L}^Y(\sum_{i \in Y} n_i \epsilon_i)$ is ample on $\mathcal{F}\ell^Y$ if and only if $n_i > 0$ for all $i \in Y$.

In the case $G = H \otimes F$ is split, the central charge map induces an isomorphism $c : \text{Pic}(\text{Gr}_H) \cong \mathbb{Z}$. We will denote \mathcal{L}_b the ample generator of the Picard group of Gr_H . Observe that, for $Y = \{i\}$ not special, the ample generator of $\text{Pic}(\mathcal{F}\ell^Y)$ has central charge a_i^\vee , which is in general greater than one. That is, the composition $\text{Pic}(\mathcal{F}\ell^Y) \rightarrow \text{Pic}(\mathcal{F}\ell) \xrightarrow{c} \mathbb{Z}$ is injective but not surjective in general.

2.3. The coherence conjecture. Now we formulate the coherence conjecture of Pappas and Rapoport. However, the originally conjecture, as stated in *loc. cit.* needs to be modified (see Remark 2.1).

Assume that G is simple, absolutely simple, simply-connected and splits over a tamely ramified extension \tilde{F}/F . Let μ be a geometrical conjugacy class of 1-parameter subgroups $(\mathbb{G}_m)_{\tilde{F}} \rightarrow G_{\text{ad}} \otimes \tilde{F}$. First assume that μ is minuscule. Let $P(\mu)$ be the corresponding maximal parabolic subgroup of H , and let $X(\mu) = H/P(\mu)$ be the corresponding partial flag variety of H . Let $\mathcal{L}(\mu)$ be the ample generator of the Picard group of $X(\mu)$. Then define

$$h_\mu(a) = \dim H^0(X(\mu), \mathcal{L}(\mu)^a).$$

If $\mu = \mu_1 + \cdots + \mu_n$ is a sum of minuscule coweights, let $h_\mu = h_{\mu_1} \cdots h_{\mu_n}$. The following is the main theorem of this paper, which is a modified version of the original coherence conjecture of Pappas and Rapoport in [PR3].

Theorem 1. *Let $\mu = \mu_1 + \cdots + \mu_n$ be a sum of minuscule coweight, then for any $Y \subset \mathbf{S}$, and ample line bundle \mathcal{L} on $\mathcal{F}\ell^Y$, we have*

$$\dim H^0(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}^a) = h_\mu(c(\mathcal{L})a),$$

where $c(\mathcal{L})$ is the central charge of \mathcal{L} .

This theorem is a consequence of the following more general theorem.

Theorem 2. *Let $\mu \in \mathbb{X}_\bullet(T_{\text{ad}})$. Then for any $Y \subset \mathbf{S}$, and ample line bundle \mathcal{L} on $\mathcal{F}\ell^Y$, we have*

$$\dim H^0(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}) = \dim H^0(\text{Gr}_{\leq \mu}, \mathcal{L}_b^{c(\mathcal{L})}).$$

Since Theorem 1 is not the same as what Pappas and Rapoport originally conjectured, and their conjecture is aimed at studying the local models of Shimura varieties, we will explain why this is the correct theorem for applications to local models in §8. Let us remark that if G is split of type A or C , Theorem 1 is proved in [PR3], using the previous results on the local models of Shimura varieties (cf. [Go1, Go2, PR2]). However, it seems that Theorem 2 is new even for symplectic groups.

One consequence of our main theorem (see §8) is that

Corollary 2.4. *The statement of Theorem 0.1 in [PR4] holds unconditionally.*

Our main theorem can be also applied to local models of other types (for example for the (even) orthogonal groups) to deduce some geometrical properties of the special fibers. This will be done in [PZ].

Remark 2.1. The original coherence conjecture in [PR3] needs to be modified. This is due to a miscalculation in [PR3, 10.a.1]. Namely, when G is simply-connected, the affine flag variety of G (denoted by \mathcal{F}_G temporarily) embeds into the affine flag variety of H (denoted by \mathcal{F}_H temporarily). Therefore there is a restriction map $\text{Pic}(\mathcal{F}_H) \rightarrow \text{Pic}(\mathcal{F}_G)$, which was described explicitly in *loc. cit.*. It turns out that there was a mistake in the calculation. However, once the calculation is restored, the coherence conjecture should be modified as the above form. Let us remark that the same miscalculation led an incorrect example in [He] Remark 19 (4), and an incorrect statement in the last sentence of the first paragraph in p. 502 of *loc. cit.* (see Proposition 4.1).

3. THE GLOBAL SCHUBERT VARIETIES

Theorem 2 will be a consequence of the geometry of the global Schubert varieties, which will be introduced in what follows. Global Schubert varieties are the function field counterparts of the local models.

3.1. The global affine Grassmannian. Let C be a smooth curve over k , and \mathcal{G} be a smooth *affine* group scheme over C . Let $\text{Gr}_{\mathcal{G}}$ be the global affine Grassmannian over C . Let us recall the functor it represents. For every k -algebra R ,

$$(3.1.1) \quad \text{Gr}_{\mathcal{G}}(R) = \left\{ (y, \mathcal{E}, \beta) \left| \begin{array}{l} y : \text{Spec} R \rightarrow C, \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } C_R, \\ \beta : \mathcal{E}|_{C_R - \Gamma_y} \cong \mathcal{E}^0|_{C_R - \Gamma_y} \text{ is a trivialization} \end{array} \right. \right\},$$

where Γ_y denotes the graph of y . This is a formally smooth ind-scheme over C .

We also have the jet group $\mathcal{L}^+ \mathcal{G}$ of \mathcal{G} . For any k -algebra R ,

$$(3.1.2) \quad \mathcal{L}^+ \mathcal{G}(R) = \left\{ (y, \beta) \left| y : \text{Spec} R \rightarrow C, \beta \in \mathcal{G}(\hat{\Gamma}_y) \right. \right\}.$$

where $\hat{\Gamma}_y$ is the formal completion of C_R along Γ_y . This is a scheme (of infinite type) formally smooth over C .

Before we proceed, let us make the following remark about $\hat{\Gamma}_y$. By definition, this is a formal affine scheme, and let A be its coordinate ring. Let $\hat{\Gamma}'_y = \text{Spec} A$. Then there are natural maps $\pi : \hat{\Gamma}_y \rightarrow C_R$ and $i : \hat{\Gamma}_y \rightarrow \hat{\Gamma}'_y$. In, [BD, §2.12], the following result is explained.

Proposition 3.1. *There is a unique map $p : \hat{\Gamma}'_y \rightarrow C_R$ such that $pi = \pi$.*

In our paper, we will soon specialize to the case $C = \mathbb{A}^1 = \text{Spec}k[v]$ so that $y : \text{Spec}R \rightarrow C$ is given by $v \mapsto y \in R$ and therefore $\Gamma_y = \text{Spec}R[v]/(v - y)$ and $\hat{\Gamma}'_y \cong \text{Spec}R[[w]]$ and the map $p : \hat{\Gamma}'_y \rightarrow C_R$ is given by $v \mapsto w + y$. Therefore, we do not need the result in *loc. cit.*. In what follows, we will not distinguish $\hat{\Gamma}_y$ and $\hat{\Gamma}'_y$ and denote both by $\hat{\Gamma}_y$ (this will not cause any problem). However, we define $\hat{\hat{\Gamma}}_y = \hat{\Gamma}'_y - \Gamma_y$. In the case $C = \mathbb{A}^1$, it is just $\text{Spec}R((w)) = \text{Spec}R[[w]][w^{-1}]$.

We can define \mathcal{LG} to be the loop group of \mathcal{G} . Let us recall that it represents the functor to associate every k -algebra R ,

$$(3.1.3) \quad \mathcal{LG}(R) = \left\{ (y, \beta) \mid y : \text{Spec}R \rightarrow C, \beta \in \mathcal{G}(\hat{\hat{\Gamma}}_y) \right\}.$$

This is a formally smooth ind-scheme over C .

Let us describe the fibers of $\mathcal{LG}, \mathcal{L}^+\mathcal{G}, \text{Gr}_{\mathcal{G}}$ over C . Let $x \in C$ be a closed point. Let \mathcal{O}_x denote the completion of the local ring of C at x and F_x be the fractional field. Then

$$(\mathcal{LG})_x \cong L(\mathcal{G}_{F_x}), \quad (\mathcal{L}^+\mathcal{G})_x \cong L^+(\mathcal{G}_{\mathcal{O}_x}), \quad (\text{Gr}_{\mathcal{G}})_x \cong \text{Gr}_{\mathcal{G}_{\mathcal{O}_x}} := L(\mathcal{G}_{F_x})/L^+(\mathcal{G}_{\mathcal{O}_x}).$$

Strictly speaking, we do not need the following remark in the sequel. However, it helps us clarify the proof of Proposition 3.5.

Remark 3.1. Let $f : \mathcal{G} \rightarrow \mathcal{G}'$ be two smooth group schemes over C such that $\mathcal{G}|_{C-\{x\}} \cong \mathcal{G}'|_{C-\{x\}}$. Then clearly the natural morphism $\mathcal{L}f : \mathcal{LG} \rightarrow \mathcal{LG}'$ will induce isomorphisms $\mathcal{LG}|_{C-\{x\}} \cong \mathcal{LG}'|_{C-\{x\}}$ and $(\mathcal{LG})_x \cong (\mathcal{LG}')_x$. However, the morphism $\mathcal{L}f$ itself is not necessarily an isomorphism.

The groups \mathcal{LG} and $\mathcal{L}^+\mathcal{G}$ naturally act on $\text{Gr}_{\mathcal{G}}$. To see this, let us use the descent lemma of Beauville-Laszlo (see [BL2], or rather a general form of this lemma given in [BD, Theorem 2.12.1]) to represent

Lemma 3.2. *The natural map*

$$\text{Gr}_{\mathcal{G}}(R) \rightarrow \left\{ (y, \mathcal{E}, \beta) \mid \begin{array}{l} y : \text{Spec}R \rightarrow C, \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on} \\ \hat{\Gamma}_y, \beta : \mathcal{E}|_{\hat{\Gamma}_y} \cong \mathcal{E}^0|_{\hat{\Gamma}_y} \text{ is a trivialization} \end{array} \right\}$$

is a bijection for each R .

Then \mathcal{LG} and $\mathcal{L}^+\mathcal{G}$ act on $\text{Gr}_{\mathcal{G}}$ by changing the trivialization β . The trivial \mathcal{G} -torsor gives $\text{Gr}_{\mathcal{G}} \rightarrow C$ a section e . Then we have the projection

$$(3.1.4) \quad \text{pr} : \mathcal{LG} \rightarrow \mathcal{LG} \cdot e = \text{Gr}_{\mathcal{G}}.$$

We need the following lemma in the sequel.

Lemma 3.3. *The formations of $\text{Gr}_{\mathcal{G}}, \mathcal{LG}, \mathcal{L}^+\mathcal{G}$ commute with any étale base change, i.e. if $f : C' \rightarrow C$ is étale, then $\text{Gr}_{\mathcal{G}} \times_C C' \cong \text{Gr}_{\mathcal{G} \times_C C'}$, etc. In addition, the action of \mathcal{LG} on $\text{Gr}_{\mathcal{G}}$ also commutes with any étale base change.*

Proof. We have the following observation. Let $y' : \text{Spec}R \rightarrow C'$ be an R -point of C' and $f(y) : \text{Spec}R \rightarrow C$ be the corresponding R -point of C . Clearly, as the formal schemes, $\hat{\Gamma}_{y'} \rightarrow \hat{\Gamma}_y$ is étale which restricts to an isomorphism along $\Gamma_{y'} \rightarrow \Gamma_y$. Therefore, $\hat{\Gamma}_{y'} \cong \hat{\Gamma}_y$ as formal schemes. In particular, their coordinate rings are isomorphic. That is $\hat{\Gamma}_{y'} \cong \hat{\Gamma}_y$ as schemes, which induces $\hat{\hat{\Gamma}}_{y'} \cong \hat{\hat{\Gamma}}_y$. The lemma clearly follows. \square

3.2. The group scheme. We will be mostly interested in the case that \mathcal{G} is a Bruhat-Tits group scheme over C . Let us specify its meaning. Let η denote the generic point of C , and for y a closed point of C , let \mathcal{O}_y denote the completion of the local ring of C at y and F_y be the fractional field. Then a smooth group scheme \mathcal{G} over C is called a Bruhat-Tits group scheme if \mathcal{G}_η is (connected) reductive, and for any closed point y of C , $\mathcal{G}_{\mathcal{O}_y}$ is a parahoric group scheme of \mathcal{G}_{F_y} .

Now let us specify the Bruhat-Tits group scheme that will be relevant to us. Let G_1 be an almost simple, absolutely simple and simply-connected, and split over a tamely ramified extension \tilde{F}/F , as in the coherent conjecture. Then we can assume that \tilde{F}/F is cyclic of order $e = 1, 2, 3$. Let γ be a generator of $\Gamma = \text{Gal}(\tilde{F}/F)$. For technical reasons, which is apparent from the statement of Theorem 2.3, we need the following well-known result.

Lemma 3.4. *There is a connected reductive group G over F , which splits over \tilde{F}/F , such that $G_{\text{der}} \cong G_1$ and $\mathbb{X}_\bullet(T) \rightarrow \mathbb{X}_\bullet(T_{\text{ad}})$ is surjective. Here T is a maximal torus of G as in §2.1.*

For example, if $G_1 = \text{SL}_n$ or Sp_{2n} , then G can be chosen as GL_n and GSp_{2n} respectively.

We let (H, T_H) be a split group with a maximal torus over \mathbb{Z} , together with an isomorphism $(G, T) \otimes_F \tilde{F} \cong (H, T_H) \otimes \tilde{F}$ as in §2.1. Let C be the chosen alcove in $A(G, S)$, and let $Y \subset \mathbf{S}$ as before. Let $[e] : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be the ramified cover given by $y \rightarrow y^e$. To distinguish these two \mathbb{A}^1 s, let us denote it as $[e] : \tilde{C} \rightarrow C$. The origin of C is denoted by 0 and the origin of \tilde{C} is denoted by $\tilde{0}$. Write $\mathring{C} = C - \{0\}$ and $\mathring{\tilde{C}} = \tilde{C} - \{\tilde{0}\}$. Observe that Γ acts on $H \times \tilde{C}$ naturally. Namely, it acts on the first factor by pinned automorphisms, and the second by transportation of structures. Let

$$\tilde{\mathcal{G}} = (\text{Res}_{\tilde{C}/C}(H \times \tilde{C}))^\Gamma.$$

Then $\tilde{\mathcal{G}}_{F_0} \cong G$ after choosing some $F_0 \cong F$. Now, replace $\tilde{\mathcal{G}}_{\mathcal{O}_0}$ by \mathcal{G}_{σ_Y} , we get a group scheme \mathcal{G} over C , satisfying

- (1) \mathcal{G}_η is connected reductive, splits over a tamely ramified extension, with connected center, such that $(\mathcal{G}_\eta)_{\text{der}}$ is simple, absolutely simple, and simply-connected;
- (2) For some choice of isomorphism $F_0 \cong F$, $\mathcal{G}_{F_0} \cong G$;
- (3) For any $y \neq 0$, $\mathcal{G}_{\mathcal{O}_y}$ is hyperspecial, (non-canonically) isomorphic to $H \otimes \mathcal{O}_y$;
- (4) $\mathcal{G}_{\mathcal{O}_0} = \mathcal{G}_{\sigma_Y}$ under the isomorphism $\mathcal{G}_{F_0} \cong G$.

Let us mention that similar group schemes have been constructed in [HNY, Ri]. For this group scheme \mathcal{G} , we know that the fiber of $\text{Gr}_{\mathcal{G}}$ over $y \neq 0$ is isomorphic to the affine Grassmannian Gr_H of H , and the fiber over 0 is isomorphic affine flag variety $\mathcal{F}\ell^Y$ of G . Likewise, the fiber of $\mathcal{L}^+\mathcal{G}$ over $y \neq 0$ is isomorphic to L^+H and the fiber over 0 is isomorphic to $L^+\mathcal{G}_{\sigma_Y}$.

Let \mathcal{T} be the subgroup scheme of \mathcal{G} , such that

- (1) \mathcal{T}_η is a maximal torus of \mathcal{G}_η ;
- (2) For any $y \neq 0$, $\mathcal{T}_{\mathcal{O}_y}$ is a split torus;
- (3) \mathcal{T}_{F_0} is the torus T and $\mathcal{T}_{\mathcal{O}_0}$ is the connected Néron model of \mathcal{T}_{F_0} .

We can construct \mathcal{T} as follows. Let

$$(3.2.1) \quad \tilde{\mathcal{T}} = (\text{Res}_{\tilde{C}/C}(T_H \times \tilde{C}))^\Gamma.$$

This is the global Néron model of \mathcal{T}_η (cf. [BLR]). Let \mathcal{T} be its neutral connected component.

In several places in the paper, we also need to extend \mathcal{G} to a Bruhat-Tits group scheme over the complete curve $\tilde{C} = C \cup \{\infty\} \cong \mathbb{P}^1$. The natural choice is taking the neutral connected component of $(\text{Res}_{\tilde{C}/\bar{C}}(H \times \tilde{C}))^\Gamma$ and modify it over \mathcal{O}_0 appropriately.

3.3. The global Schubert variety. It turns out that it is more convenient to base change everything over C to \tilde{C} . Let u (resp. v) denote a global coordinate of \tilde{C} (resp. C) such that the map $[e] : \tilde{C} \rightarrow C$ is given by $u \mapsto v^e$. Recall that $0 \in C(k)$ (resp. $\tilde{0} \in \tilde{C}(k)$) is given by $v = 0$ (resp. $u = 0$). The crucial step toward the construction of the global Schubert varieties is the following proposition.

Proposition 3.5. *For each $\mu \in \mathbb{X}_\bullet(\mathcal{T}_\eta) \cong \mathbb{X}_\bullet(T_H)$, there is a section*

$$s_\mu : \tilde{C} \rightarrow \mathcal{LT} \times_C \tilde{C}$$

such that for any $\tilde{y} \in \tilde{C}(k)$ the element

$$s_\mu(\tilde{y}) \in (\mathcal{LT})_{\tilde{y}}(k) = \mathcal{T}_{F_y}(F_y), \quad y = [e](\tilde{y})$$

maps under the Kottwitz homomorphism $\kappa : \mathcal{T}_{F_y}(F_y) \rightarrow \mathbb{X}_\bullet(\mathcal{T}_\eta)_{\text{Gal}(F_y^s/F_y)}$ to the image of μ under the natural projection $\mathbb{X}_\bullet(\mathcal{T}_\eta) \rightarrow \mathbb{X}_\bullet(\mathcal{T}_\eta)_{\text{Gal}(F_y^s/F_y)}$.

The proposition is obvious for split groups. But for the ramified groups, the proof is a little bit complicated and will not be used in the main body of the paper. Therefore, those who are only interested in split groups can skip the proof.

Proof. Let us first review how to construct an element in $t_\mu \in T(k((t)))$ whose image under the Kottwitz homomorphism (2.1.3) is λ under the map $\mathbb{X}_\bullet(T) \rightarrow \mathbb{X}_\bullet(T)_\Gamma$. Let $k((s))/k((t))$ be a finite separable extension of degree n so that $T_{k((s))}$ splits, where $s^n = t$. Then $\lambda(s) \in T(k((s)))$. By the construction of the Kottwitz homomorphism, we can take t_λ to be the image of $\lambda(s)$ under the norm map $T(k((s))) \rightarrow T(k((t)))$.

Now we construct s_μ . Let $\tilde{\mathcal{T}}$ is as in (3.2.1). This is a global Néron model. We will first construct a section $s_\lambda : \tilde{C} \rightarrow \mathcal{LT}$ and then prove it indeed factors as $s_\mu : \tilde{C} \rightarrow \mathcal{LT} \rightarrow \mathcal{LT}$.

Let $\Gamma_{[e]}$ denote the graph of $[e] : \tilde{C} \rightarrow C$. By definition,

$$\text{Hom}_C(\tilde{C}, \mathcal{LT}) = \text{Hom}_C(\mathring{\Gamma}_{[e]}, \tilde{\mathcal{T}}) = \text{Hom}(\mathring{\Gamma}_{[e]} \times_C \tilde{C}, T_H)^\Gamma,$$

where Γ acts on $\mathring{\Gamma}_{[e]} \times_C \tilde{C}$ via the action on the second factor.

Recall that we have the global coordinates u, v and the map $[e] : \tilde{C} \rightarrow C$ is given by $v \mapsto u^e$. Then $\mathcal{O}_{\mathring{\Gamma}_{[e]}} \cong k[u]((v - u^e))$. Therefore, the ring of functions on $\mathring{\Gamma}_{[e]} \times_C \tilde{C}$ can be written as

$$A = k[u_1]((v - u_1^e)) \otimes_{k[v]} k[u_2],$$

where the map $k[v] \rightarrow k[u_2]$ is given by $v \mapsto u_2^e$. Let γ be a generator of $\Gamma = \text{Aut}(\tilde{C}/C)$ acting on u_2 as $u_2 \mapsto \xi u_2$, where ξ is a primitive e 'th root of unit. For $i = 1, \dots, e$, the element $(\xi^i \otimes u_2 - u_1 \otimes 1)$ is invertible in A , and therefore gives a morphism

$$x_i : \mathring{\Gamma}_{[e]} \times_C \tilde{C} \rightarrow \mathbb{G}_m.$$

Clearly $x_i \circ \gamma = x_{i+1}$ (as usual, $x_{i+e} = x_i$).

Now choose a basis $\omega_1, \dots, \omega_\ell$ of $\mathbb{X}_\bullet(T_H)$. Let us define

$$s_\mu : \mathring{\Gamma}_{[e]} \times_C \tilde{C} \rightarrow T_H$$

as

$$\omega_j(s_\mu) = x_1^{(\mu, \gamma \omega_j)} x_2^{(\mu, \gamma^2 \omega_j)} \dots x_e^{(\mu, \gamma^e \omega_j)}.$$

Clearly, s_μ is independent of the choice of $\omega_1, \dots, \omega_\ell$ (however, it depends on the global coordinate u on \tilde{C}). Furthermore, s_μ is Γ -equivariant. Therefore, we constructed a section $s_\mu : \tilde{C} \rightarrow \mathcal{LT}$.

Now we prove that this section indeed factors as $s_\mu : \tilde{C} \rightarrow \mathcal{LT} \rightarrow \mathcal{LT}$. In other words, the morphism $\mathring{\Gamma}_{[e]} \rightarrow \tilde{\mathcal{T}}$ factors as $\mathring{\Gamma}_{[e]} \rightarrow \mathcal{T} \rightarrow \tilde{\mathcal{T}}$. By definition, \mathcal{T} is the neutral connected component of $\tilde{\mathcal{T}}$. Therefore, it is enough to prove that the image of $\mathring{\Gamma}_{[e]}|_0 \rightarrow \tilde{\mathcal{T}}|_0$ lands in the neutral connected component of $\tilde{\mathcal{T}}|_0$. Observe that $\mathring{\Gamma}_{[e]}|_0 \cong \text{Speck}((u_1))$. Let \tilde{C}_0 be the fiber of $\tilde{C} \rightarrow C$ over 0 so that $\tilde{C}_0 \cong k[u]/u^e$ with a Γ -action. It has a unique closed point $\tilde{0}$. Recall that $\tilde{T}|_0 = (\text{Res}_{\tilde{C}_0/k}(T_H \times \tilde{C}_0))^\Gamma$ and therefore, there is a canonical map $\epsilon : \tilde{\mathcal{T}}|_0 \rightarrow T_H^\Gamma$ given by adjunction, making the following diagram commute

$$\begin{array}{ccccc} \text{Hom}_C(\mathring{\Gamma}_{[e]}, \tilde{\mathcal{T}}) & \longrightarrow & \text{Hom}(\mathring{\Gamma}_{[e]}|_0, \tilde{\mathcal{T}}|_0) & \xrightarrow{\epsilon} & \text{Hom}(\mathring{\Gamma}_{[e]}|_0, T_H^\Gamma) \\ \parallel & & \parallel & & \parallel \\ \text{Hom}(\mathring{\Gamma}_{[e]} \times_C \tilde{C}, T_H)^\Gamma & \longrightarrow & \text{Hom}(\mathring{\Gamma}_{[e]}|_0 \times \tilde{C}_0, T_H)^\Gamma & \longrightarrow & \text{Hom}(\mathring{\Gamma}_{[e]}|_0 \times \{\tilde{0}\}, T_H^\Gamma) \end{array}$$

In our case $\epsilon(s_\mu) : \mathring{\Gamma}_{[e]}|_0 \rightarrow T_H$ is given by

$$\omega_j(\epsilon(s_\mu)) = (-u_1)^{(\sum_{\gamma \in \Gamma} \gamma \mu, \omega_j)}.$$

In other words, $\epsilon(s_\mu)$ is the composition

$$\mathring{\Gamma}_{[e]}|_0 \xrightarrow{-u_1} \mathbb{G}_m \xrightarrow{\sum_{\gamma \in \Gamma} \gamma \mu} T_H.$$

Since for any Γ -invariant coweight μ , the image $\mu : \mathbb{G}_m \rightarrow T_H^\Gamma$ lands in the neutral connected component of T_H^Γ (the torus part), $s_\mu : \tilde{C} \rightarrow \mathcal{LT}$ factors through $\tilde{C} \rightarrow \mathcal{LT} \rightarrow \mathcal{LT}$.

Finally, let us check that $s_\mu : \tilde{C} \rightarrow \mathcal{LT} \times_C \tilde{C}$ satisfies the desired properties as claimed in the proposition.

Let $\tilde{y} \in \tilde{C}(k)$ be a closed point given by $u \mapsto \tilde{y} \in k$. Then $s_\mu(\tilde{y})$ corresponds to $s_\mu(\tilde{y}) : \text{Speck}((v - y^e)) \otimes_{k[v]} k[u_2] \rightarrow T_H$ given by

$$\omega_j(s_\mu(\tilde{y})) = \prod_{i=1}^e (\xi^i 1 \otimes u_2 - y)^{(\mu, \gamma^i \omega_j)}.$$

If $\tilde{y} = 0$, the assertion of the proposition follows directly from the review of the construction of t_μ at the beginning. If $\tilde{y} \neq 0$, let $w = 1 \otimes u_2 - y$. Then

$$\prod_{i=1}^e (\xi^i 1 \otimes u_2 - y)^{(\mu, \gamma^i \omega_j)} = w^{(\mu, \omega_j)} f(w)$$

where

$$f(w) = \prod_{i=1}^{e-1} (\xi^i 1 \otimes u_2 - y)^{(\mu, \gamma^i \omega_j)} \in k[[w]]^\times.$$

Therefore, as an element in $T_H(k((w)))$, which is canonically isomorphic to $\mathcal{LT}_{\tilde{y}}$, $s_\mu(\tilde{y})$ maps to μ under the Kottwitz homomorphism. \square

Remark 3.2. For general μ , there is no such section $C \rightarrow \mathcal{LT}$ satisfying the property of the proposition. This is the reason that we want to base change everything over C to \tilde{C} . However, if $\mu \in \mathbb{X}_\bullet(T)$ is defined over F , then s_μ indeed descends to a section $C \rightarrow \mathcal{LT}$. This means that in this case the variety $\overline{\text{Gr}}_{\mathcal{G},\mu}$ defined below, which a priori is a variety over \tilde{C} , descends to a variety over C . One can summarize this by saying that $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is defined over the "reflex field" of μ . The same phenomenon appears in the theory of Shimura varieties.

The composition of s_μ and the natural morphism (see (3.1.4)) $\text{pr} : \mathcal{LT} \rightarrow \text{Gr}_{\mathcal{T}}$ (resp. $\mathcal{LT} \rightarrow \mathcal{LG}$) gives a section $\tilde{C} \rightarrow \text{Gr}_{\mathcal{T}} \times_C \tilde{C}$ (resp. $\tilde{C} \rightarrow \mathcal{LG} \times_C \tilde{C}$), which is still denoted by s_μ .

The construction of $\tilde{C} \rightarrow \mathcal{LT} \times_C \tilde{C}$ will depend on the choice of the global coordinate u of \tilde{C} , but the section $s_\mu : \tilde{C} \rightarrow \text{Gr}_{\mathcal{T}} \times_C \tilde{C}$ does not. Indeed, there is the following moduli interpretation of such section. Recall that $\text{Gr}_{\mathcal{T}}$ is ind-proper over C ([He]), and therefore, s_μ is uniquely determined by a section $\tilde{C} \rightarrow \text{Gr}_{\mathcal{T}} \times_C \tilde{C} \cong \text{Gr}_{T_H \times \tilde{C}}^\circ$ (by Lemma 3.3). Then this section, under the moduli interpretation of $\text{Gr}_{T_H \times \tilde{C}}^\circ$, is given as follows: let Δ be the diagonal of \tilde{C}^2 , and $\mathcal{O}_{\tilde{C}^2}(\mu\Delta)$ be the T_H -torsor on \tilde{C}^2 , such that for any weight ν of T_H , the associated line bundle is $\mathcal{O}_{\tilde{C}^2}((\mu, \nu)\Delta)$. This T_H -torsor has a canonical trivialization away from Δ .

Lemma 3.6. *The map $s_\mu : \tilde{C} \rightarrow \text{Gr}_{\mathcal{T}}$ corresponds to (\mathcal{E}, β) , where \mathcal{E} is the T_H -torsor $\mathcal{O}_{\tilde{C}^2}(\mu\Delta)$, and β is its canonical trivialization over $\tilde{C}^2 - \Delta$.*

Proof. The Kottwitz homomorphism $\kappa : LT_H(k) \rightarrow \mathbb{X}_\bullet(T_H)$ induces an isomorphism $\text{Gr}_H(k) \cong LT_H(k)/L^+T_H(k)$. On the other hand, recall that if we fix a point x on the curve \tilde{C} , we can interpret Gr_{T_H} as the set of (\mathcal{E}, β) , where \mathcal{E} is an T_H -torsor and β is a trivialization of \mathcal{E} away from x . Under this interpretation, any $t_\mu \in \mathbb{X}_\bullet(T_H)$ is interpreted as the T_H -torsor $\mathcal{O}_{\tilde{C}}(\mu x)^5$, with its canonical trivialization away from x . Then the lemma is clear. \square

Under the natural morphism $\text{Gr}_{\mathcal{T}} \rightarrow \text{Gr}_{\mathcal{G}}$, we obtain a section of $\text{Gr}_{\mathcal{G}} \times_C \tilde{C}$, still denoted by s_μ .

Notation. In what follows, we denote $\text{Gr}_{\mathcal{G}} \times_C \tilde{C}$ (resp. $\mathcal{L}^+\mathcal{G} \times_C \tilde{C}$, resp. $\mathcal{LG} \times_C \tilde{C}$) by $\widetilde{\text{Gr}}_{\mathcal{G}}$ (resp. $\widetilde{\mathcal{L}^+\mathcal{G}}$, resp. $\widetilde{\mathcal{LG}}$).

Definition 3.1. For each $\mu \in \mathbb{X}_\bullet(\mathcal{T}_\eta) \cong \mathbb{X}_\bullet(T_H)$, the global Schubert variety $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is the minimal $\widetilde{\mathcal{L}^+\mathcal{G}}$ -stable irreducible closed subvariety of $\widetilde{\text{Gr}}_{\mathcal{G}}$ that contains s_μ .

Let us emphasize that $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is not a subvariety of $\text{Gr}_{\mathcal{G}}$. Rather, it lies in $\text{Gr}_{\mathcal{G}} \times_C \tilde{C}$. Recall that for any $\mu \in \mathbb{X}_\bullet(T_{\text{ad}})$, one defines a subset $\text{Adm}(\mu) \subset \widetilde{W}$ as in (2.1.6). The main geometry of $\overline{\text{Gr}}_{\mathcal{G},\mu}$ we will prove in this paper is

Theorem 3. *Let y be a closed point of \tilde{C} . Then*

$$(\overline{\text{Gr}}_{\mathcal{G},\mu})_y \cong \begin{cases} \bigcup_{w \in \text{Adm}^Y(\mu)} \mathcal{F}\ell_w^Y & y = \tilde{0} \\ \overline{\text{Gr}}_\mu & y \neq \tilde{0}. \end{cases}$$

In particular, all the fibers are reduced.

⁵The reason that t_μ represents $\mathcal{O}_{\tilde{C}}(\mu x)$ rather than $\mathcal{O}_{\tilde{C}}(-\mu x)$ is due to the original sign convention of the Kottwitz homomorphism in [Ko].

We first prove the easy part of the theorem.

Lemma 3.7. $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_y \cong \overline{\mathrm{Gr}}_\mu$ for $y \neq \tilde{0}$.

Proof. Write $\mathring{\tilde{C}} = \tilde{C} - \tilde{0}$. We want to show that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\mathring{\tilde{C}}}$ is isomorphic to $\overline{\mathrm{Gr}}_\mu \times \mathring{\tilde{C}}$. First we have a canonical isomorphism

$$(3.3.1) \quad \mathcal{G} \times_C \mathring{\tilde{C}} \cong H \times \mathring{\tilde{C}}$$

and therefore by Lemma 3.3

$$(3.3.2) \quad \mathrm{Gr}_{\mathcal{G}} \times_C \mathring{\tilde{C}} \cong \mathrm{Gr}_{H \times \mathring{\tilde{C}}}, \quad \mathcal{L}_{\mathcal{G}} \times_C \mathring{\tilde{C}} \cong \mathcal{L}(H \times \mathring{\tilde{C}}).$$

Secondly, $\mathring{\tilde{C}} \cong \mathbb{G}_m$ which admits a global coordinate u so that $\mathcal{L}(H \times \mathring{\tilde{C}}) \cong LH \times \mathring{\tilde{C}}$ and $\mathrm{Gr}_{H \times \mathring{\tilde{C}}} \cong \mathrm{Gr}_H \times \mathring{\tilde{C}}$. Finally, by Lemma 3.6, the section $s_\mu : \mathring{\tilde{C}} \rightarrow \mathrm{Gr}_{\mathcal{G}} \times_C \mathring{\tilde{C}} \cong \mathrm{Gr}_H \times \mathring{\tilde{C}}$ satisfying $s_\mu(\mathring{\tilde{C}}) \subset \overline{\mathrm{Gr}}_\mu \times \mathring{\tilde{C}}$. \square

From this lemma, it is clear that we can make the following convention.

Convention. When we discuss $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, we will assume that $\mu \in \mathbb{X}_\bullet(T_H)$ is dominant with respect to the chosen Borel.

At this moment, we can also see that

Lemma 3.8. *The scheme $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}} \subset (\mathrm{Gr}_{\mathcal{G}})_0 \cong \mathcal{F}\ell^Y$ contains $\mathcal{F}\ell_w^Y$ for $w \in \mathrm{Adm}^Y(\mu)$.*

Proof. Clearly, it is enough to show that $\mathcal{F}\ell_\lambda^Y \subset (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ for any λ in Λ , where Λ is the W_0 -orbit in $\mathbb{X}_\bullet(T)_\Gamma$ containing μ as constructed in §2.1. Observe that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is the flat closure of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\mathring{\tilde{C}}}$ in $\widehat{\mathrm{Gr}}_{\mathcal{G}}$, since the later is clearly $\widehat{\mathcal{L}^+ \mathcal{G}}$ -stable. Then the claims follows from that for any $\lambda \in \mathbb{X}_\bullet(T_H)$ in the \bar{W} -orbit of μ , $s_\lambda(\tilde{0}) \in \mathcal{F}\ell_\lambda^Y$ and $s_\lambda(\mathring{\tilde{C}}) \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\mathring{\tilde{C}}} \cong \overline{\mathrm{Gr}}_\mu \times \mathring{\tilde{C}}$. \square

To prove the theorem, it is remains to show that

Theorem 3.9. *The underlying reduced subscheme of $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ is $\bigcup_{w \in \mathrm{Adm}^Y(\mu)} \mathcal{F}\ell_w^Y$.*

Theorem 3.10. *$(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ is reduced.*

Let us ask the following question which we could not answer: is the variety $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ Cohen-Macaulay? If this is true, it will also imply that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is normal.

4. LINE BUNDLES ON $\mathrm{Gr}_{\mathcal{G}}$ AND $\mathrm{Bun}_{\mathcal{G}}$

This subsection explains why Theorem 3 and Theorem 2 are equivalent to each other. The key ingredients are the line bundles on the global affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$. Observe that $\mathrm{Gr}_{\mathcal{G}}$ can be disconnected. This will create some complications to determine the line bundle on $\mathrm{Gr}_{\mathcal{G}}$ directly. Instead, we will pass to the group scheme \mathcal{G}' , whose generic fiber is simply-connected so that we can use the results of Heinloth [He] directly.

4.1. Line bundles on $\mathrm{Gr}_{\mathcal{G}}$ and $\mathrm{Bun}_{\mathcal{G}}$. In this subsection, we temporary assume that C is a smooth curve over k and \mathcal{G} is a Bruhat-Tits group scheme over C such that \mathcal{G}_{η} is almost simple, absolutely simple, and simply-connected.

Proposition 4.1. *Let \mathcal{L} be a line bundle on $\mathrm{Gr}_{\mathcal{G}}$, then the function $c_{\mathcal{L}}$ that associates every $y \in C(k)$ the central charge of the restriction of \mathcal{L} to $(\mathrm{Gr}_{\mathcal{G}})_y$ is constant.*

This proposition implies that the statement in the last sentence of the first paragraph in p. 502 of [He] is not correct.

Proof. Let $\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}/C)$ denote the relative sheaf of Picard groups over C . As explained in [He], this is an étale sheaf over C . Let $D = \mathrm{Ram}(\mathcal{G})$ be the set of points of C such that for every $y \in \mathrm{Ram}(\mathcal{G})$, the fiber \mathcal{G}_y is not semisimple. This is a finite set. Then there is a short exact sequence

$$(4.1.1) \quad 1 \rightarrow \prod_{y \in D} \mathbb{X}^{\bullet}(\mathcal{G}_y) \rightarrow \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}/C) \rightarrow \mathfrak{c} \rightarrow 1,$$

where \mathfrak{c} is a constructible sheaf, with all fibers isomorphic to \mathbb{Z} and is constant on $C - D$.

According to the description of the sheaf \mathfrak{c} in Remark 19 (3) of *loc. cit.*, if \mathcal{L} is a line bundle on $\mathrm{Gr}_{\mathcal{G}}$ such that $c_{\mathcal{L}}(y) = 0$ for some $y \in C(k)$, then $c_{\mathcal{L}} = 0$. Therefore, to prove the proposition, It is enough to construct one line bundle \mathcal{L}_{2c} on $\mathrm{Gr}_{\mathcal{G}}$, such that $c_{\mathcal{L}_{2c}}$ is constant on C .

Let $\mathcal{V}_0 = \mathrm{Lie}\mathcal{G}$ be the Lie algebra of \mathcal{G} . This is a locally free \mathcal{O}_C -modules on C of rank $\dim_{\eta} \mathcal{G}_{\eta}$, on which \mathcal{G} acts by adjoint representation. This induces a morphism $\mathcal{G} \rightarrow \mathrm{GL}(\mathcal{V}_0)$, and therefore a morphism $i : \mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$. Let $\mathcal{L}_{\mathrm{det}}$ denote the determinant line bundle on $\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$. Let us recall its construction. Namely, we want to associated every $\mathrm{Spec} R \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$ a line bundle on $\mathrm{Spec} R$ in a compatible way. Recall a morphism $\mathrm{Spec} R \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$ represents a morphism $y \in C(R)$, a vector bundle \mathcal{V} on C_R and an isomorphism $\mathcal{V}|_{C_R - \Gamma_y} \cong \mathcal{V}_0|_{C_R - \Gamma_y}$. There exists some N large enough such that

$$\mathcal{V}_0(-N\Gamma_y) \subset \mathcal{V} \subset \mathcal{V}_0(N\Gamma_y)$$

and $\mathcal{V}_0(N\Gamma_y)/\mathcal{V}$ is V -flat. Then the line bundle on V is

$$\det(\mathcal{V}_0(N\Gamma_y)/\mathcal{V}) \otimes \det(\mathcal{V}_0(N\Gamma_y)/\mathcal{V}_0)^{-1},$$

which is independent of the choice of N up to a canonical isomorphism.

The pullback $i^* \mathcal{L}_{\mathrm{det}}$ is a line bundle on $\mathrm{Gr}_{\mathcal{G}}$, which will be our \mathcal{L}_{2c} . To see this is the desired line bundle, we need to calculate its central charge when restricted to each $y \in C(k)$. Let $D = \mathrm{Ram}(\mathcal{G})$. First consider $y \in C - D$. Then the map $i_y : (\mathrm{Gr}_{\mathcal{G}})_y \rightarrow (\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)})_y$ is just

$$\mathrm{Gr}_H \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie} H)},$$

where H is the split Chevalley group over \mathbb{Z} such that $G \otimes k(\eta)^s \cong H \otimes k(\eta)^s$. It is well known that in this case $i_y^* \mathcal{L}_{\mathrm{det}}$ over y has central charge $2h^{\vee}$, where h^{\vee} is the dual Coxeter number of H (in fact, this statement is a consequence of the following argument).

It remains to calculate the central charge of \mathcal{L}_{2c} over $y \in D$. Without loss of generality, we can assume that D consists of one point, denoted by 0. Let $G = \mathcal{G}_{F_0}$. So let $y = 0$. Then the closed embedding $i_0 : (\mathrm{Gr}_{\mathcal{G}})_0 \rightarrow (\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)})_0$ is just

$$LG/L^+ \mathcal{G}_{\mathcal{O}_0} \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie} \mathcal{G}_{\mathcal{O}_0})}.$$

Let us first assume that $\mathcal{G}_{\mathcal{O}_0}$ is an Iwahori group scheme of G_{F_0} . Write $B = \mathcal{G}_{\mathcal{O}_0}$ and $\mathcal{F}\ell = LG/L^+B$ as usual. We claim that in this case

Lemma 4.2. $i_0^* \mathcal{L}_{\det} \cong \mathcal{L}(2 \sum_{i \in \mathbf{S}} \epsilon_i)$.

Assuming this equation, we find the central charge of $i_0^* \mathcal{L}_{\det}$ is $2 \sum_{i \in \mathbf{S}} a_i^\vee$. By checking all the affine Dynkin diagrams, we find that

$$\sum_{i \in \mathbf{S}} a_i^\vee = h^\vee.$$

In fact, we find for affine Dynkin diagrams $X_N^{(r)}$, where $X = A, B, C, D, E, F, G$ and $r = 1, 2, 3$, the sum $\sum a_i^\vee$ is independent of r (see [Kac, Remark 6.1]), and it is well-known (or by definition) that for $r = 1$, $\sum a_i^\vee = h^\vee$. Therefore, the proposition follows in this case.

Now we prove Lemma 4.2. This is equivalent to prove that the restriction of $i_0^* \mathcal{L}_{\det}$ to each \mathbb{P}_j^1 (whose definition is given in §2.2) is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$. Let us give the moduli interpretations of the morphism $\mathcal{F}\ell \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie} B)}$. Recall $\mathcal{F}\ell$ classifies the Iwahori group schemes of G . Then $\mathcal{F}\ell \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie} B)}$ maps an Iwahori group scheme B' of G to its Lie algebra $\mathrm{Lie} B'$, which is a free \mathcal{O}_0 -module, together with the canonical isomorphism $\mathrm{Lie} B' \otimes F_0 \cong \mathrm{Lie} G \cong \mathrm{Lie} B \otimes F_0$.

For $j \in \mathbf{S}$, let P_j be the minimal parahoric (but not Iwahori) group scheme corresponding to j . Then the subscheme $\mathbb{P}_j^1 \subset \mathcal{F}\ell$ classifies the Iwahori group schemes of G that map to P_j . Let $P_j^u \rightarrow P_j$ be the "unipotent radical" of P_j . More precisely, P_j^u is smooth over \mathcal{O}_0 with $P_j^u \otimes F_0 = G$ and the special fiber of P_j^u maps onto the unipotent radical of the special fiber of P_j . If B' is an Iwahori group scheme of G that maps to P_j , then

$$\mathrm{Lie} P_j^u \subset \mathrm{Lie} B' \subset \mathrm{Lie} P_j.$$

Let \bar{P}_j^{red} be the reductive quotient of the special fiber of P_j , then \bar{P}_j^{red} is isomorphic to $\mathrm{GL}_2, \mathrm{SL}_2$ or SO_3 over k . Let $\mathbb{G}(2, \mathrm{Lie} \bar{P}_j^{\mathrm{red}}) \cong \mathbb{P}^2$ denote the Grassmannian (over k) of 2-planes in the three dimensional vector space $\mathrm{Lie} \bar{P}_j^{\mathrm{red}}$. We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}_j^1 & \longrightarrow & \mathbb{G}(2, \mathrm{Lie} \bar{P}_j^{\mathrm{red}}) \\ \downarrow & & \downarrow \\ \mathcal{F}\ell & \longrightarrow & \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie} B)} \end{array}$$

where $\mathbb{P}_j^1 \rightarrow \mathbb{G}(2, \mathrm{Lie} \bar{P}_j^{\mathrm{red}})$ is given by

$$B' \mapsto (\mathrm{Lie} B' / \mathrm{Lie} P_j^u \subset \mathrm{Lie} P_j / \mathrm{Lie} P_j^u \cong \mathrm{Lie} \bar{P}_j^{\mathrm{red}})$$

and $\mathbb{G}(2, \mathrm{Lie} \bar{P}_j^{\mathrm{red}}) \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie} B)}$ is given by realizing that $\mathbb{G}(2, \mathrm{Lie} \bar{P}_j^{\mathrm{red}})$ represents the free \mathcal{O}_0 -modules that are in between $\mathrm{Lie} P_j^u$ and $\mathrm{Lie} P_j$.

By construction, the restriction of \mathcal{L}_{\det} to $\mathbb{G}(2, \mathrm{Lie} \bar{P}_j^{\mathrm{red}})$ is the (positive) determinant line bundle on $\mathbb{G}(2, \mathrm{Lie} \bar{P}_j^{\mathrm{red}})$, or $\mathcal{O}_{\mathbb{P}^2}(1)$. Therefore, the restriction of \mathcal{L}_{\det} to \mathbb{P}_j^1 is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$. This finishes the proof of Lemma 4.2 and therefore the proposition in the case $\mathcal{G}_{\mathcal{O}_0}$ is Iwahori.

Now let $\mathcal{G}_{\mathcal{O}_0}$ be a general parahoric group scheme. Let \mathcal{G}' be the group scheme over C together with $\mathcal{G}' \rightarrow \mathcal{G}$ which is an isomorphism over $C - \{0\}$ and $\mathcal{G}'_{\mathcal{O}_0}$ is Iwahori. Let $\mathcal{V}_0 = \mathrm{Lie} \mathcal{G}$ and $\mathcal{V}'_0 = \mathrm{Lie} \mathcal{G}'$. We have the natural map

$$p : (\mathrm{Gr} \mathcal{G}')_0 \rightarrow (\mathrm{Gr} \mathcal{G})_0$$

induced from $\mathcal{G}' \rightarrow \mathcal{G}$ and the maps

$$i : \mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}, \quad i' : \mathrm{Gr}_{\mathcal{G}'} \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}'_0)}.$$

We need to show that $p^*i_0^*\mathcal{L}_{\det}$ and $i_0'^*\mathcal{L}_{\det}$ have the same central charge (observe that these two line bundles are not isomorphic). From this, we conclude that the central charge of $i^*\mathcal{L}_{\det}$ is also constant along C .

Let us extend \mathcal{G} and \mathcal{G}' to group schemes over the complete curve \bar{C} such that $\mathcal{G}|_{\bar{C}-\{0\}} = \mathcal{G}'|_{\bar{C}-\{0\}}$. Let $\mathrm{Bun}_{\mathcal{G}}$ (resp. $\mathrm{Bun}_{\mathcal{G}'}$) be the moduli stack of \mathcal{G} -torsors (\mathcal{G}' -torsors) on \bar{C} . Let $\mathcal{G}_0, \mathcal{G}'_0$ be the restriction of the two group schemes over $0 \in C$, and let P be the image of $\mathcal{G}'_0 \rightarrow \mathcal{G}_0$. This is indeed a Borel subgroup of \mathcal{G}'_0 . Recall that by restricting a \mathcal{G}' -torsor to $0 \in C$, we obtain a map $(\mathrm{Gr}_{\mathcal{G}'})_0 \xrightarrow{r} \mathbb{B}\mathcal{G}'_0$, and we have the similar map for \mathcal{G} . Then we have the following diagram with both squares Cartesian

$$(4.1.2) \quad \begin{array}{ccccccc} (\mathrm{Gr}_{\mathcal{G}'})_0 & \longrightarrow & \mathrm{Bun}_{\mathcal{G}'} & \xrightarrow{r} & \mathbb{B}\mathcal{G}'_0 & \longrightarrow & \mathbb{B}P \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\mathrm{Gr}_{\mathcal{G}})_0 & \longrightarrow & \mathrm{Bun}_{\mathcal{G}} & \xrightarrow{r} & \mathbb{B}\mathcal{G}'_0 & & \end{array}$$

Indeed, it is clear that the left square is Cartesian because $\mathcal{G}|_{\bar{C}-\{0\}} = \mathcal{G}'|_{\bar{C}-\{0\}}$. The fact that the second square is Cartesian is established in Proposition 9.1.

Let $y : \mathrm{Spec} R \rightarrow (\mathrm{Gr}_{\mathcal{G}'})_0$ be a morphism given by (\mathcal{E}, β) , where \mathcal{E} is a \mathcal{G}' -torsor on C_R . Then we have the natural short exact sequence

$$0 \rightarrow \mathrm{ad} \mathcal{E} \rightarrow \mathrm{ad}(\mathcal{E} \times^{\mathcal{G}'} \mathcal{G}) \rightarrow \mathcal{E} \times^{\mathcal{G}'} (\mathrm{Lie} \mathcal{G} / \mathrm{Lie} \mathcal{G}') \rightarrow 0.$$

On the other hand, $p : (\mathrm{Gr}_{\mathcal{G}'})_0 \rightarrow (\mathrm{Gr}_{\mathcal{G}})_0$ is relatively smooth morphism since $\mathbb{B}P \rightarrow \mathbb{B}\mathcal{G}'_0$ is smooth. Let \mathcal{T}_p denote the relative tangent sheaf. We claim that $\mathcal{E} \times^{\mathcal{G}'} (\mathrm{Lie} \mathcal{G} / \mathrm{Lie} \mathcal{G}') \cong y^* \mathcal{T}_p$, where $y^* \mathcal{T}_p$ is the sheaf on $\mathrm{Spec} R$, regarded as a sheaf on C_R via the closed embedding $\{0\} \times \mathrm{Spec} R =: \{0\}_R \rightarrow C_R$. But this follows from (4.1.2) and

$$\mathcal{E} \times^{\mathcal{G}'} (\mathrm{Lie} \mathcal{G} / \mathrm{Lie} \mathcal{G}') \cong \mathcal{E}|_{\{0\}_R} \times^{\mathcal{G}'_0} (\mathrm{Lie} \mathcal{G}_0 / \mathrm{Lie} P) \cong (\mathcal{E}|_{\{0\}_R} \times^{\mathcal{G}'_0} P) \times^P (\mathrm{Lie} \mathcal{G}_0 / \mathrm{Lie} P).$$

Therefore, we have

$$(4.1.3) \quad 0 \rightarrow \mathrm{ad} \mathcal{E} \rightarrow \mathrm{ad}(\mathcal{E} \times^{\mathcal{G}''} \mathcal{G}') \rightarrow y^* \mathcal{T}_p \rightarrow 0,$$

Let us finish the prove that $p^*i_0^*\mathcal{L}_{\det}$ and $i_0'^*\mathcal{L}_{\det}$ have the same central charge and therefore the proposition. From the above lemma,

$$(4.1.4) \quad p^*i_0^*\mathcal{L}_{\det} \cong i_0'^*\mathcal{L}_{\det} \otimes \det(\mathcal{T}_p).$$

So it is enough to prove that $\det \mathcal{T}_p$ as a line bundle on $(\mathrm{Gr}_{\mathcal{G}'})_0$ has central charge zero. But from (4.1.2), $\det \mathcal{T}_p$ is a pullback of some line bundle from $\mathbb{B}P$, and hence from $\mathbb{B}\mathcal{G}'_0$, which has zero central charge by (2.2.3). \square

Now, we assume that C is a complete curve and let $\mathrm{Bun}_{\mathcal{G}}$ be the moduli stack of \mathcal{G} -torsors on C . Let $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$ be the Picard group of rigidified line bundles (trivialized over the trivial \mathcal{G} -torsor) on $\mathrm{Bun}_{\mathcal{G}}$. Let $D = \mathrm{Ram}(\mathcal{G})$. Observe that $\prod_{y \in C(k)} \mathbb{X}^{\bullet}(\mathcal{G}_y) = \prod_{y \in D} \mathbb{X}^{\bullet}(\mathcal{G}_y)$. Fix $0 \in C(k)$. Let $\mathcal{F}\ell^Y = L\mathcal{G}_{F_0}/L^+\mathcal{G}_{\mathcal{O}_0}$, which is a partial affine flag variety of \mathcal{G}_{F_0} . According to [He], we have the following

commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \prod_{y \in C(k)} \mathbb{X}^\bullet(\mathcal{G}_y) & \longrightarrow & \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{X}^\bullet(\mathcal{G}_0) & \longrightarrow & \mathrm{Pic}(\mathcal{F}\ell^Y) & \xrightarrow{c} & \mathbb{Z} \longrightarrow 0
\end{array}$$

The left vertical arrow is the projection to the factor corresponding to 0 and the right vertical arrow is injective (but not necessarily surjective). Probably, one can show that $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}) \rightarrow \mathbb{Z}$ is in fact given by $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}) \rightarrow \Gamma(C, \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}/C)) \rightarrow \Gamma(C, \mathfrak{c}) \cong \mathbb{Z}$ and the right vertical arrow is the natural restriction map $\Gamma(C, \mathfrak{c}) \rightarrow \mathfrak{c}|_0$. We do not need this fact. What we need is that from the above diagram, for any $\mathcal{L} \in \mathrm{Pic}(\mathcal{F}\ell^Y)$, a certain tensor power of it will descend to a line bundle on $\mathrm{Bun}_{\mathcal{G}}$. Therefore we conclude

Corollary 4.3. *Let C be a smooth but (not necessarily complete) curve and \mathcal{G} be a Bruhat-Tits group scheme over C such that \mathcal{G}_η is almost simple, absolutely simple and simply-connected. Let H be the split Chevalley group over \mathbb{Z} such that $\mathcal{G} \otimes k(\eta)^s \cong H \otimes k(\eta)^s$. Let $0 \in C(k)$ and let \mathcal{L} be a line bundle on $\mathcal{F}\ell^Y = L\mathcal{G}_{F_0}/L^+\mathcal{G}_{\mathcal{O}_0}$. Then there is a line bundle on $\mathrm{Gr}_{\mathcal{G}}$, whose restriction to $(\mathrm{Gr}_{\mathcal{G}})_0 \cong \mathcal{F}\ell^Y$ is isomorphic to \mathcal{L}^n for some $n \geq 1$, and whose restriction to $(\mathrm{Gr}_{\mathcal{G}})_y \cong \mathrm{Gr}_H(y \notin \mathrm{Ram}(\mathcal{G}))$ is isomorphic to $\mathcal{L}_b^{nc(\mathcal{L})}$.*

Proof. Let \bar{C} be complete curve containing C . We extend \mathcal{G} to a Bruhat-Tits group scheme over \bar{C} . Then some tensor power \mathcal{L}^n of \mathcal{L} descends to a line bundle \mathcal{L}' on $\mathrm{Bun}_{\mathcal{G}}$. Let $h_{\mathrm{glob}} : \mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Bun}_{\mathcal{G}}$ be the natural projection. Then $h_{\mathrm{glob}}^* \mathcal{L}'$ is a line bundle on $\mathrm{Gr}_{\mathcal{G}}$ whose restriction to $(\mathrm{Gr}_{\mathcal{G}})_0$ is isomorphic to \mathcal{L}^n , and whose restriction to $(\mathrm{Gr}_{\mathcal{G}})_y \cong \mathrm{Gr}_H$ ($y \notin \mathrm{Ram}(\mathcal{G})$) has central charge $c(\mathcal{L}^n)$, and therefore is isomorphic to $\mathcal{L}_b^{nc(\mathcal{L})}$. \square

4.2. Theorem 3 is equivalent to Theorem 2. Let us begin with a general construction. Let \mathcal{G} be a Bruhat-Tits group scheme over a curve C . Then away from a finite subset $D \subset C$, $\mathcal{G}|_{C-D}$ is reductive. Let $\mathcal{G}_{\mathrm{der}}|_{C-D}$ be the derived group of $\mathcal{G}|_{C-D}$ so that for $y \in C(k)$, $(\mathcal{G}_{\mathrm{der}})_{F_y}$ is the derived group of \mathcal{G}_{F_y} . It is known that there is a canonical bijection between the facets in the building of $(\mathcal{G}_{\mathrm{der}})_{F_y}$ and those in the building of \mathcal{G}_{F_y} , and under this bijection, the corresponding parahoric group scheme for $(\mathcal{G}_{\mathrm{der}})_{F_y}$ maps to the corresponding parahoric group scheme for \mathcal{G}_{F_y} . Therefore, we can extend $\mathcal{G}_{\mathrm{der}}|_{C-D}$ to a Bruhat-Tits group scheme over C together with a morphism $\mathcal{G}_{\mathrm{der}} \rightarrow \mathcal{G}$, such that for all $y \in D$, $(\mathcal{G}_{\mathrm{der}})_{\mathcal{O}_y} \rightarrow \mathcal{G}_{\mathcal{O}_y}$ is the morphism of parahoric group schemes given by the facet determined by $\mathcal{G}_{\mathcal{O}_y}$.

Definition 4.1. The group scheme $\mathcal{G}_{\mathrm{der}}$ together with the morphism $\mathcal{G}_{\mathrm{der}} \rightarrow \mathcal{G}$ is called the derived group of \mathcal{G} .

Now let us specialize the group \mathcal{G} to be the Bruhat-Tits group scheme over $C = \mathbb{A}^1$ as defined in §3.2. Let us denote $\mathcal{G}' = \mathcal{G}_{\mathrm{der}}$ for simplicity. Let $\bar{C} = C - \{0\}$. Observe that $\mathcal{G}'_{\bar{C}}$ is reductive and $\mathcal{G}'_{F_0} \cong G_1 = G_{\mathrm{der}}$ and for $y \neq 0$, $\mathcal{G}'_{\mathcal{O}_y}$ is hyperspecial for $H_{\mathrm{der}} \otimes \mathcal{O}_y$. In addition, \mathcal{G}'_η is simply-connected.

Let us explain why Theorem 3 and Theorem 2 are equivalent. The natural morphism $\mathcal{G}' \rightarrow \mathcal{G}$ induces a morphism $\mathrm{Gr}_{\mathcal{G}'} \rightarrow \mathrm{Gr}_{\mathcal{G}}$. One can show that this is a closed embedding (we will not use this fact but this can be also seen from the following reasoning). But at least it follows directly from [PR3, §6] that both $(\mathrm{Gr}_{\mathcal{G}'})_0 \rightarrow (\mathrm{Gr}_{\mathcal{G}})_0$ and $\mathrm{Gr}_{\mathcal{G}'}|_{\bar{C}} \rightarrow \mathrm{Gr}_{\mathcal{G}}|_{\bar{C}}$ are closed embeddings, which induce isomorphisms from the

formers to the reduced subschemes of the neutral connected components of the latter. Let $\mu \in \mathbb{X}_\bullet(T)$, and let $\overline{\text{Gr}}_{\mathcal{G},\mu}$ be the corresponding global Schubert variety as in §3.3. Recall the section s_μ from Proposition 3.5. Regard it as a section of $\widetilde{\mathcal{LG}}$, which acts on $\widetilde{\text{Gr}}_{\mathcal{G}}$. Then

$$s_\mu^{-1}\overline{\text{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}} \subset \widetilde{\text{Gr}}_{\mathcal{G}'}|_{\tilde{C}}.$$

This follows from $t_\mu^{-1}\overline{\text{Gr}}_\lambda \subset \text{Gr}_{H_{\text{der}}}$ for any $\mu \in \mathbb{X}_\bullet(T_H)$, where t_μ is considered as any lifting of $t_\mu \in \widetilde{W}$ to $T_H(F)$. Let $\text{Gr}_{\mathcal{G}',\leq\mu}$ be the flat closure of $s_\mu^{-1}\overline{\text{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}}$ in $\widetilde{\text{Gr}}_{\mathcal{G}'}$. We have the natural map

$$\text{Gr}_{\mathcal{G}',\leq\mu} \rightarrow s_\mu^{-1}\overline{\text{Gr}}_{\mathcal{G},\mu},$$

which induces a closed embedding $(\text{Gr}_{\mathcal{G}',\leq\mu})_{\tilde{0}} \rightarrow (s_\mu^{-1}\overline{\text{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ since $\mathcal{F}\ell_{\text{sc}}^Y \rightarrow \mathcal{F}\ell^Y$ is a closed embedding. By flatness, this necessarily implies that $(\text{Gr}_{\mathcal{G}',\leq\mu})_{\tilde{0}} \cong (s_\mu^{-1}\overline{\text{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$.

Let τ_μ be the image of μ in $\Omega \cong \mathbb{X}_\bullet(T)_\Gamma/\mathbb{X}_\bullet(T_{\text{sc}})_\Gamma$ and let $Y^\circ \subset \mathbf{S}$ so that $\sigma_{Y^\circ} = \tau_\mu^{-1}(\sigma_Y)$ as before. Let $g \in G_1(F)$ be a lifting of $t_{-\mu}\tau_\mu \in W_{\text{aff}}$. Then since $\mathcal{F}\ell_w^Y \in (\overline{\text{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ for $w \in \text{Adm}^Y(\mu)$ (see Lemma 3.8), $g(Y^\circ \mathcal{F}\ell_{\text{sc},w}^Y) \subset (\text{Gr}_{\mathcal{G}',\leq\mu})_{\tilde{0}}$ for $w \in \text{Adm}^Y(\mu)^\circ$. In other words, $\mathcal{A}^Y(\mu)^\circ \subset (\text{Gr}_{\mathcal{G}',\leq\mu})_{\tilde{0}}$.

Let \mathcal{L} be an ample line bundle on $\mathcal{F}\ell_{\text{sc}}^Y$, and we prolong its certain tensor power to a line bundle on $\text{Gr}_{\mathcal{G}'}$ by Theorem 4.3. Then we have

$$\dim \Gamma((\text{Gr}_{\mathcal{G}',\leq\mu})_y, \mathcal{L}_b^{nc(\mathcal{L})}) = \dim \Gamma((\text{Gr}_{\mathcal{G}',\leq\mu})_{\tilde{0}}, \mathcal{L}^n) \geq \dim \Gamma(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}^n)$$

by the flatness and the fact that $H^1(Y \mathcal{F}\ell_w^{Y'}, \mathcal{L}) = 0$ for any Schubert variety $Y \mathcal{F}\ell_w^{Y'}$ and any ample line bundle \mathcal{L} . In addition, the equality holds if and only if $\mathcal{A}^Y(\mu)^\circ = (\text{Gr}_{\mathcal{G}',\leq\mu})_{\tilde{0}}$. Clearly, for $y \neq \tilde{0}$, $(\text{Gr}_{\mathcal{G}',\leq\mu})_y = g\text{Gr}_{\leq\mu}$ and therefore

$$\Gamma((\text{Gr}_{\mathcal{G}',\leq\mu})_y, \mathcal{L}_b^{nc(\mathcal{L})}) \cong \Gamma(\text{Gr}_{\leq\mu}, \mathcal{L}_b^{c(\mathcal{L})})$$

by Theorem 4.3. Therefore, Theorem 2 implies Theorem 3. Conversely, Theorem 3 implies that the statement of Theorem 2 holds for $\mathcal{L}^n, \mathcal{L}^{2n}, \dots$ and therefore holds for \mathcal{L} .

5. SOME PROPERTIES OF $\overline{\text{Gr}}_{\mathcal{G},\mu}$

In this section, we study two basic geometrical structures of $\overline{\text{Gr}}_{\mathcal{G},\mu}$: (i) in §5.2, we will construct certain affine charts of $\overline{\text{Gr}}_{\mathcal{G},\mu}$, which turn out to be isomorphic to affine spaces over \tilde{C} ; and (ii) in §5.3, we will construct a \mathbb{G}_m -action on $\overline{\text{Gr}}_{\mathcal{G},\mu}$, so that the map $\overline{\text{Gr}}_{\mathcal{G},\mu} \rightarrow \tilde{C}$ is \mathbb{G}_m -equivariant, where \mathbb{G}_m acts on $\tilde{C} = \mathbb{A}^1$ by natural dilatation. To establish (i), we will need to first construct the global root subgroups of \mathcal{LG} as in §5.1. We shall remark that all the results for G being split are obvious. It is for the ramified group G that some complicated discussions are needed. Those who are only interested in split groups can skip this section.

5.1. global root groups. We shall introduce the "root subgroups" of \mathcal{LG} , whose fibers over $0 \in C$ is the usual root subgroups of the loop group LG as constructed in [PR3, 9.a,9.b].

Let us first review the shape of root groups of G . Let (H, B_H, T_H, X) be a pinned Chevalley group over \mathbb{Z} as in §2.1. Let Ξ be the group of pinned automorphisms of H (more precisely, the group of pinned automorphisms of H_{der} , which is simple, almost simple, simply-connected by our assumption). So Ξ is a cyclic group of order 1, 2 or 3. Let $\tilde{\Phi} = \Phi(H, T_H)$ be the set of roots of H with respect to T_H . For each

$\tilde{a} \in \Phi(H, T_H)$, let $\tilde{U}_{\tilde{a}}$ denote the corresponding root group. Then for each $\gamma \in \Xi$, one has an isomorphism $\gamma : \tilde{U}_{\tilde{a}} \cong \tilde{U}_{\gamma\tilde{a}}$. The stabilizer of \tilde{a} in Ξ is either trivial or the whole group. Let us choose a Chevalley-Steinberg system of H , i.e. for each $\tilde{a} \in \Phi(H, T_H)$, an isomorphism $x_{\tilde{a}} : \mathbb{G}_a \cong \tilde{U}_{\tilde{a}}$ over \mathbb{Z} . In addition, we require that:

- (1) if $\tilde{a} \in \Delta$ is a simple root, then $X_{\tilde{a}} = dx_{\tilde{a}}(1)$, where $X = \sum_{\tilde{a} \in \Delta} X_{\tilde{a}}$;
- (2) if the stabilizer of \tilde{a} in Ξ is trivial, then $\gamma \circ x_{\tilde{a}} = x_{\gamma\tilde{a}}$ for any $\gamma \in \Xi$.

Clearly, such a system exists. Note that if the stabilizer of \tilde{a} is the whole group Ξ , it is not necessarily true that one can make $\gamma \circ x_{\tilde{a}} = x_{\tilde{a}}$, as can be seen for SL_3 . In this case, one obtains a quadratic character

$$(5.1.1) \quad \chi_{\tilde{a}} : \Xi \rightarrow \mathrm{Aut}_{\mathbb{Z}}(\mathbb{G}_a) = \{\pm 1\}$$

such that $\gamma \circ x_{\tilde{a}} = x_{\tilde{a}} \circ \chi_{\tilde{a}}(\gamma)$. Of course, this can happen only if the order of Ξ is 2.

Recall that $\Gamma = \mathrm{Aut}(\tilde{C}/C)$ is a group of order $e = 1, 2, 3$, which acts on H via pinned automorphisms and the corresponding map $\Gamma \rightarrow \Xi$ is injective.

Let $j : \Phi(H, T_H) \rightarrow \Phi(G, S)$ be the restriction of the root systems. For $a \in \Phi(G, S)$, let

$$\eta(a) = \{\tilde{a} \in \Phi(H_{K'}, T_{K'}) \mid j(\tilde{a}) = ma, m \geq 0\}.$$

Denote $\tilde{U}_{\eta(a)}$ be the subgroup of H generated by $\tilde{U}_{\tilde{a}}, \tilde{a} \in \eta(a)$. This is a subgroup of H invariant under Ξ . As a scheme, $\tilde{U}_{\eta(a)} \cong \prod_{\tilde{a} \in \eta(a)} \tilde{U}_{\tilde{a}}$, where the product is taken over any given order on $\eta(a)$.

Let

$$U_{a,C} = (\mathrm{Res}_{\tilde{C}/C} \tilde{U}_{\eta(a), \tilde{C}})^{\Gamma}.$$

Then $U_{a,C} \otimes F_0$ is the root group of $\mathcal{G}_{F_0} \cong G$ corresponding to a . For $y \neq 0$, $(U_{a,C})_y \cong \tilde{U}_{\eta(a)}$. In addition, $\mathcal{L}U_{a,C}$ is a subgroup of $\mathcal{L}\mathcal{G}$.

Recall that we fixed the special vertex v_0 at the beginning. Using this vertex, we identify $A(G, S)$ with $\mathbb{X}_{\bullet}(S)_{\mathbb{R}}$ and we can write affine roots as $a + m$, where $a \in \Phi(G, S)$ and $m \in \frac{1}{e}\mathbb{Z}$ (the valuation is normalized so that u has valuation $1/e$). Let us construct for each affine root $a + m$, a closed immersion

$$(5.1.2) \quad x_{a+m} : \mathbb{G}_{a,C} \rightarrow \mathcal{L}U_{a,C}.$$

Let us describe of x_{a+m} at the level of R -points. Recall we write $C = \mathrm{Spec}k[v], \tilde{C} = \mathrm{Spec}k[u]$, such that $[e] : \tilde{C} \rightarrow C$ is given by $v \mapsto u^e$. Let R be a k -algebra and let $y : \mathrm{Spec}R \rightarrow C$ be an R -point of C . We identify $\mathrm{Hom}_C(\mathrm{Spec}R, \mathbb{G}_{a,C})$ with R in an obvious manner. We thus need to construct a map (functorial with respect to R)

$$x_{a+m} : R \rightarrow \mathrm{Hom}_C(\mathrm{Spec}R, \mathcal{L}U_{a,C}).$$

The graph of $y : \mathrm{Spec}R \rightarrow C$ is $\Gamma_y = \mathrm{Spec}R[v]/(v - y)$ and $\hat{\Gamma}_y = \mathrm{Spec}R((v - y))$. Now, by definition

$$\mathrm{Hom}_C(\mathrm{Spec}R, \mathcal{L}U_{a,C}) = \mathrm{Hom}(\mathrm{Spec}R((v - y)) \times_C \tilde{C}, \tilde{U}_{\eta(a)})^{\Gamma},$$

where Γ acts on $\mathrm{Spec}R((v - y)) \times_C \tilde{C}$ via the action on \tilde{C} , and acts on $\tilde{U}_{\eta(a)}$ via the pinned isomorphisms.

Let us make the following notation. Each element

$$s \in R((v - y)) \otimes_{k[v]} k[u]$$

determines a morphism $\mathrm{Spec}R((v - y)) \times_C \tilde{C} \rightarrow \mathbb{G}_a$, and let $x_{\tilde{a}}(s) : \mathrm{Spec}R((v - y)) \times_C \tilde{C} \rightarrow \tilde{U}_{\tilde{a}}$ denote the composition of this morphism with $x_{\tilde{a}} : \mathbb{G}_a \rightarrow \tilde{U}_{\tilde{a}}$.

Now we construction x_{a+m} . Write $m = m_1 + \frac{m_2}{e}$ where $m_1, m_2 \in \mathbb{Z}, 0 \leq m_2 < e$. There are two cases.

(i) $2a \notin \Phi(G, S)$. In this case, Γ acts transitively on $\eta(a)$. There are two subcases.

(ia) $\eta(a) = \tilde{a}$, so that Γ fixes \tilde{a} and $\tilde{U}_{\eta(a)} = \tilde{U}_{\tilde{a}}$. Define

$$x_{a+m}(r) = x_{\tilde{a}}(r(v-y)^{m_1} \otimes u^{m_2}).$$

Since $a+m$ is an affine root, Γ acts on u^{m_2} exactly via the quadratic character $\chi_{\tilde{a}}$ as defined in (5.1.1), $x_{a+m}(r) \in \text{Hom}(\text{Spec}R((v-y)) \times_C \tilde{C}, \tilde{U}_{\eta(a)})^\Gamma$.

(ib) Γ acts simply transitively on $\eta(a)$. Pick up $\tilde{a} \in \eta(a)$ and $\gamma \in \Gamma$ a generator. Using the isomorphism $\prod_{i=1}^e \tilde{U}_{\gamma^i(\tilde{a})} \cong \tilde{U}_{\eta(a)}$, one defines

$$x_{a+m}(r) = \prod_{i=1}^e x_{\gamma^i(\tilde{a})}(r(v-y)^{m_1} \otimes \gamma^i(u)^{m_2}).$$

Since for $\tilde{a}, \tilde{a}' \in \eta(a)$, the groups $\tilde{U}_{\tilde{a}}$ and $\tilde{U}_{\tilde{a}'}$ commute, we still have $x_{a+m}(r) \in \text{Hom}(\text{Spec}R((v-y)) \times_C \tilde{C}, \tilde{U}_{\eta(a)})^\Gamma$.

(ii) $2a \in \Phi(G, S)$, so that $\eta(a) = \{\tilde{a}, \tilde{a}', \tilde{a} + \tilde{a}'\}$. In this case, $\text{char } k \neq 2$, $e = 2, m_2 = 1$, and the quadratic charcter $\chi_{\tilde{a}+\tilde{a}'}$ is non-trivial. Recall that for any s, s' ,

$$(5.1.3) \quad x_{\tilde{a}}(s)x_{\tilde{a}'}(s') = x_{\tilde{a}'}(s')x_{\tilde{a}}(s)x_{\tilde{a}+\tilde{a}'}(\pm ss'),$$

where \pm depends on $x_{\tilde{a}}, x_{\tilde{a}'}, x_{\tilde{a}+\tilde{a}'}$, but not on s, s' . Define

$$x_{a+m}(r) = x_{\tilde{a}}(r(v-y)^{m_1} \otimes u)x_{\tilde{a}'}(-r(v-y)^{m_1} \otimes u)x_{\tilde{a}+\tilde{a}'}(\mp \frac{1}{2}r^2(v-y)^{2m_1} \otimes u^2)$$

where \mp is the sign **opposite** the sign \pm in (5.1.3). Using (5.1.3), it is clear that $x_{a+m}(r) \in \text{Hom}(\text{Spec}R((v-y)) \times_C \tilde{C}, \tilde{U}_{\eta(a)})^\Gamma$.

We have completed the construction of (5.1.2). Observe that over $0 \in C$ (i.e. by setting $y = 0$), the map (5.1.2) reduces to an isomorphism of \mathbb{G}_a and the root subgroup of LG corresponding to $a+m$, as constructed in [PR3, 9.a, 9.b]. This motives us to define

Definition 5.1. Let $a+m$ be an affine root of G . The subgroup scheme $\mathcal{U}_{a+m} = x_{a+m}(\mathbb{G}_{a,C})$ is called root subgroup of \mathcal{LG} corresponding to $a+m$.

Remark 5.1. By taking the fibers $U_{a+m} = (\mathcal{U}_{a+m})_0 \subset (\mathcal{LG})_0 \cong LG$, we obtain the root subgroups of LG . If we do not identify $A(G, S)$ with $\mathbb{X}_\bullet(S)_\mathbb{R}$ via v_0 , we write them as U_α , where α is an affine root.

The following a few lemmas about the root subgroups for (global) loop groups are the counterparts of some well-known results about the root subgroups of Kac-Moody groups.

Lemma 5.1. For $a+m, b+n$ ($a \notin \mathbb{R}b$) two affine roots of G , the commutator $(\mathcal{U}_{a+m}, \mathcal{U}_{b+n})$ is contained in the group generated by $\mathcal{U}_{(pa+qb)+(pm+qn)}$, where $p, q \in \mathbb{Z}_{>0}$ such that $(pa+qb) + (pm+qn)$ is also an affine root of G .

Proof. Let R be a k -algebra. It is enough to check that for any $y \in C(R)$, the commutator $(\mathcal{U}_{a+m}(R), \mathcal{U}_{b+n}(R))$ is contained in the group generated by those $\mathcal{U}_{(pa+qb)+(pm+qn)}(R)$ where $p, q \in \mathbb{Z}_{>0}$ and $(pa+qb) + (pm+qn)$ is an affine root.

Let $\Phi^{nd} \subset \Phi = \Phi(G, S)$ denote the set of non-divisible roots, i.e. $a \in \Phi^{nd}$ if $a/2 \notin \Phi$. Let us define $\Psi_{a,b} \subset \tilde{\Phi} = \Phi(H, T_H)$ be the set of roots containing those

$$\Psi_{a,b} = \{\tilde{a} \in \tilde{\Phi} \mid j(\tilde{a}) = pa + qb \text{ for } p, q \in \mathbb{Z}_{>0}\} = \bigcup_{pa+qb \in \Phi^{nd}, p, q > 0} \eta(pa + qb),$$

where $j : \tilde{\Phi} \rightarrow \Phi$. For $\tilde{a} \in \Psi_{a,b}$ such that $j(\tilde{a}) = pa + qb$, let $k(\tilde{a}) = pm + qn$. Using the same notation as in §5.1, let us define

$$\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})} \subset \mathcal{L}(\text{Res}_{\tilde{C}/C} \tilde{U}_{\tilde{a},\tilde{C}})$$

be the group over C , whose R -points over $y : \text{Spec} R \rightarrow C$ are given by

$$\{x_{\tilde{a}}(r(v-y)^{k(\tilde{a})_1} \otimes u^{k(\tilde{a})_2}), r \in R\} \subset \text{Hom}_C(\text{Spec} R, \mathcal{L}(\text{Res}_{\tilde{C}/C} \tilde{U}_{\tilde{a},\tilde{C}})),$$

where $k(\tilde{a})_1, k(\tilde{a})_2 \in \mathbb{Z}$, such that $0 \leq k(\tilde{a})_2 < e$ and $k(\tilde{a})_1 + \frac{k(\tilde{a})_2}{e} = k(\tilde{a})$. Let $p, q \in \mathbb{Z}_{>0}$, and let $\tilde{\mathcal{U}}_{\eta(pa+qb), pm+qn}$ be the group generated by $\tilde{\mathcal{U}}_{\tilde{a}, k(\tilde{a})}$ for those $\tilde{a} \in \eta(pa+qb) \subset \Psi_{a,b}$. Clearly, over $y \in C(R)$,

$$\mathcal{U}_{(pa+qb)+(pm+qn)}(R) = \tilde{\mathcal{U}}_{\eta(pa+qb), pm+qn}(R) \cap \mathcal{L}U_{pa+qb, C}(R).$$

Let $\tilde{\mathcal{U}}_{\Psi_{a,b}, m, n}$ be the group generated by $\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}$, $\tilde{a} \in \Psi_{a,b}$. Recall that for the fixed Chevalley-Steinberg system $\{x_{\tilde{a}}, \tilde{a} \in \tilde{\Phi}\}$, and for two roots $\tilde{a}, \tilde{b} \in \tilde{\Phi}$, there exists $c(p, q) \in \mathbb{Z}$ for $p, q \in \mathbb{Z}_{>0}$ such that for any ring R and $r, s \in R$, the commutator $(x_{\tilde{a}}(r), x_{\tilde{b}}(s))$ can be written as $(x_{\tilde{a}}(r), x_{\tilde{b}}(s)) = \prod_{p\tilde{a}+q\tilde{b} \in \tilde{\Phi}, p, q > 0} x_{p\tilde{a}+q\tilde{b}}(c(p, q)r^p s^q)$. Therefore, the commutator of $(\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}, \tilde{\mathcal{U}}_{\tilde{b}+k(\tilde{b})})$ is contained in the group generated by $\tilde{\mathcal{U}}_{p\tilde{a}+q\tilde{b}+k(p\tilde{a}+q\tilde{b})}$, where $p, q \in \mathbb{Z}_{>0}$ and $p\tilde{a}+q\tilde{b} \in \tilde{\Phi}$. Therefore, by [BT1, Proposition 6.1.6] (applying to the case $Y_{\tilde{a}} = \tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}(R)$ for $y \in C(R)$), we have

$$\tilde{\mathcal{U}}_{\Psi_{a,b}, m, n}(R) \xleftarrow{\sim} \prod_{\tilde{a} \in \Psi} \tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}(R) \xrightarrow{\sim} \prod_{pa+qb \in \Phi^{nd}, p, q > 0} \tilde{\mathcal{U}}_{\eta(pa+qb), pm+qn}(R).$$

Next, let $\mathcal{L}U_{(a,b)}$ be the group generated by $\mathcal{L}U_{pa+qb, C}$, $pa+qb \in \Phi, p, q \in \mathbb{Z}_{>0}$. Again by *loc.cit.*, for $y \in C(R)$, there is a bijection

$$\prod_{pa+qb \in \Phi^{nd}, p, q > 0} \mathcal{L}U_{pa+qb, C}(R) \xrightarrow{\sim} \mathcal{L}U_{(a,b)}(R).$$

Combining the above two isomorphisms, we thus obtain that

$$\begin{aligned} & (\mathcal{U}_{a+m}(R), \mathcal{U}_{b+n}(R)) \subset \tilde{\mathcal{U}}_{\Psi_{a,b}, m, n} \cap \mathcal{L}U_{(a,b)}(R) \\ &= \prod_{pa+qb \in \Phi^{nd}, p, q > 0} (\tilde{\mathcal{U}}_{\eta(pa+qb), pm+qn}(R) \cap \mathcal{L}U_{pa+qb, C}(R)) \\ &= \prod_{pa+qb \in \Phi^{nd}, p, q > 0} \mathcal{U}_{(pa+qb)+(pm+qn)}(R). \end{aligned}$$

□

Recall that the pinning (2.1.1) determines a set of positive roots $\Phi(G, S)^+$. By our choice of the alcove C , $U_{a,C} \subset \mathcal{G}$ for $a \in \Phi(G, S)^+$. Therefore, $\mathcal{U}_{a+m} \subset \mathcal{L}^+ \mathcal{G}$ for $a \in \Phi(G, S)^+, m \geq 0$. Let $\mu \in \mathbb{X}_{\bullet}(T)_{\Gamma}^+$, and let

$$(5.1.4) \quad \Phi_{\mu} = \{a + m \mid a + m \text{ is an affine root}, a \in \Phi(G, S)^+, 0 \leq m < (\lambda, a)\}.$$

Let $\mathcal{U}_{\Phi_{\mu}}$ be the subgroup of $\mathcal{L}^+ \mathcal{G}$ generated by $\mathcal{U}_{a+m}, a + m \in \Phi_{\mu}$.

Lemma 5.2. *For some given order (which will be specified in the proof) on the set Φ_{μ} , the natural map*

$$\prod_{a+m \in \Phi_{\mu}} \mathcal{U}_{a+m} \rightarrow \mathcal{U}_{\Phi_{\mu}}$$

is an isomorphism of schemes.

Proof. Again, it is enough to prove the isomorphism at the level of R -points. By Lemma 5.1, the collection of groups $\{\prod_m \mathcal{U}_{a+m}(R), a \in \Phi^+\}$ satisfies the condition as required by [BT1, Proposition 6.1.7]. Fix an order on $\Phi^+ \cap \Phi^{nd}$, we can extend it to an order on Φ^+ by requiring if $a, 2a \in \Phi^+$, then $a < 2a < b$ for any $b \in \Phi^{nd}$ such that $a < b$. In addition, we can give an order on Φ_λ by requiring $a + m < b + n$ if either $a < b$ or $a = b, m < n$. With this given order, the lemma follows by *loc. cit.*. \square

5.2. Some affine charts of $\overline{\text{Gr}}_{\mathcal{G}, \mu}$. We introduce certain affine charts of $\overline{\text{Gr}}_{\mathcal{G}, \mu}$, which turn out to be isomorphic to affine spaces. First let $\mu \in \mathbb{X}_\bullet(T)_\Gamma^+$. Recall the section $s_\mu : \tilde{C} \rightarrow \widetilde{\mathcal{LG}}$ as constructed in the paragraph after Proposition 3.5. Consider

$$c_\mu : \mathcal{U}_{\Phi_\mu} \times_C \tilde{C} \rightarrow \overline{\text{Gr}}_{\mathcal{G}, \mu}, \quad g \mapsto gs_\mu.$$

Lemma 5.3. *The morphism c_μ is an open immersion.*

Proof. Observe that $\dim \mathcal{U}_{\Phi_\mu} = (2\rho, \mu) + 1$, which is the same as the dimension of $\overline{\text{Gr}}_{\mathcal{G}, \mu}$. Therefore, it is enough to prove that the stabilizer of s_μ in \mathcal{U}_{Φ_μ} is trivial. Recall that \mathcal{LG} acts on $\text{Gr}_{\mathcal{G}}$, and the stabilizer of the section $e : C \rightarrow \text{Gr}_{\mathcal{G}}$ (defined by the trivial \mathcal{G} -torsor) is $\mathcal{L}^+\mathcal{G}$. Therefore, the stabilizer in $\widetilde{\mathcal{LG}}$ of the section s_μ is $s_\mu(\widetilde{\mathcal{L}^+\mathcal{G}})s_\mu^{-1}$. Therefore, it is enough to prove that $\widetilde{\mathcal{L}^+\mathcal{G}} \cap s_\mu^{-1}(\mathcal{U}_{\Phi_\mu} \times_C \tilde{C})s_\mu$ is trivial, or equivalently $\mathcal{L}^+U_{a,C} \times_C \tilde{C} \cap s_\mu^{-1}(\mathcal{U}_{a+m} \times_C \tilde{C})s_\mu$ is trivial for all $a + m \in \Phi_\mu$.

Let us analyze the R -points of $s_\mu^{-1}(\mathcal{U}_{a+m} \times_C \tilde{C})s_\mu$ over $y : \text{Spec} R \rightarrow \tilde{C}$. Recall that $s_\mu(y)$ is given by the Γ -equivariant map

$$s_\mu(y) : \text{Spec} R((v - y^e)) \otimes_{k[v]} k[u_2] \rightarrow T_H$$

such that for any weight ω of T_H , the composition $\omega s_\mu(y)$ (which is determined by an invertible element in $R((v - y^e)) \otimes_{k[v]} k[u_2]$) is $\prod_{i=1}^e (1 \otimes \gamma^i(u_2) - y \otimes 1)^{(\mu, \gamma^i \omega)}$. Let us choose ω such that $(\mu, \gamma^i \omega) \leq 0$ for $i = 1, 2, \dots, e$. If we write

$$\prod_{i=1}^e (1 \otimes \gamma^i(u_2) - y \otimes 1)^{(\mu, \gamma^i \omega)} = \sum r_{pq} (v - y^e)^p \otimes u_2^q$$

with $r_{pq} \in R, 0 \leq q < e$, and $r_{pq} = 0$ for $p \ll 0$, then the smallest p such that $r_{pq} \neq 0$ is $\leq \lfloor \frac{(\sum \gamma^i \mu, \omega)}{e} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer that is $\leq x$. Now we set $\omega = -\tilde{a}$ with $j(\tilde{a}) = a \in \Phi(G, S)^+$. Then for $m = m_1 + \frac{m_2}{e} < (\mu, a)$ with $m_1, m_2 \in \mathbb{Z}_{\geq 0}, m_2 < e$, we have

$$\prod_{i=1}^e (1 \otimes \gamma^i(u_2) - y \otimes 1)^{(-\mu, \gamma^i(\tilde{a}))} ((v - y^e)^{m_1} \otimes u_2^{m_2}) \notin R[[v - y^e]] \otimes_{k[v]} k[u_2],$$

because $m_1 \leq m < (\mu, a) \leq -\lfloor -(\mu, a) \rfloor = -\lfloor \frac{(\sum \gamma^i \mu, -\tilde{a})}{e} \rfloor$.

Clearly, this implies that $\mathcal{L}^+U_{a,C} \times_C \tilde{C} \cap s_\mu^{-1}(\mathcal{U}_{a+m} \times_C \tilde{C})s_\mu$ is the identity for all $a + m \in \Phi_\mu$ and therefore the lemma. \square

Let $\Lambda = W_0\mu \subset \mathbb{X}_\bullet(T)_\Gamma$ as in §2.1. Let $\lambda \in \Lambda$. We can construct the affine chart of $\overline{\text{Gr}}_{\mathcal{G}, \mu}$ corresponding to λ as follows. Choose $w \in W_0$ such that $w\mu = \lambda$. Let $\Phi_\lambda = w(\Phi_\mu)$. One can define the group $\mathcal{U}_{\Phi_\lambda}$ correspondingly, which is isomorphic to an affine space of dimension $(2\rho, \lambda) + 1$ as schemes. Recall that $s_\lambda \subset \overline{\text{Gr}}_{\mathcal{G}, \mu}$ and define

$$c_\lambda : \mathcal{U}_{\Phi_\lambda} \times_C \tilde{C} \rightarrow \overline{\text{Gr}}_{\mathcal{G}, \mu}, \quad g \mapsto gs_\lambda.$$

By the same argument as above, c_λ is an open immersion.

In what follows, we denote the image of c_λ ($\lambda \in \Lambda$) by U_λ , so that U_λ is affine open in $\overline{\text{Gr}}_{\mathcal{G},\mu}$ which is smooth over \tilde{C} (indeed an affine space over \tilde{C}).

5.3. A \mathbb{G}_m -action on $\overline{\text{Gr}}_{\mathcal{G},\mu}$. Let \mathcal{G}_{v_0} is the special parahoric group scheme of G as in (2.1.2). In this subsection, we assume that the group scheme \mathcal{G} constructed in §3.2 over $C = \text{Spec} k[v]$ satisfies the following condition: after identifying $F_0 = k((v)) = F = k((t))$ so that we identify $\mathcal{G}_{F_0} = G$, there exists a morphism $\mathcal{G}_{\mathcal{O}_0} \rightarrow \mathcal{G}_{v_0}$. In other words, after identifying the building of \mathcal{G}_{F_0} with G , then the closure of the facet determined by $\mathcal{G}_{\mathcal{O}_0}$ contains the special vertex v_0 corresponds to \mathcal{G}_{v_0} .

Let $f : \widetilde{\text{Gr}}_{\mathcal{G}} \rightarrow \tilde{C}$ be the structural map. We construct a natural \mathbb{G}_m -action on $\widetilde{\text{Gr}}_{\mathcal{G}}$, which lifts the natural action of \mathbb{G}_m on \tilde{C} via dilatations. In addition, each $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is stable under this \mathbb{G}_m -action.

The construction of the \mathbb{G}_m -action on $\widetilde{\text{Gr}}_{\mathcal{G}}$ is straightforward. Recall that the global coordinate on \tilde{C} is u and on C is v , and that the map $[e] : \tilde{C} \rightarrow C$ is given by $v \mapsto u^e$. Recall that an R -point of $\widetilde{\text{Gr}}_{\mathcal{G}}$ is given by $u \mapsto y$ and a \mathcal{G} -torsor \mathcal{E} on C_R , which is trivialized over $C_R - \Gamma_{[e](y)}$. Let $r \in R^\times$ be an R -point of \mathbb{G}_m . We need to construct a new \mathcal{G} -torsor on C_R , together with a trivialization over $C_R - \Gamma_{[e](ry)}$. Indeed, let $r^e : C_R \rightarrow C_R$ given by $v \mapsto r^e v$. It maps $\Gamma_{[e](y)}$ to $\Gamma_{[e](ry)}$. Then the pullback of \mathcal{E} along r^{-e} is an $(r^{-e})^*\mathcal{G}$ -torsor on C_R , together with a trivialization on $C_R - \Gamma_{[e](ry)}$. Therefore, to complete the construction, it is enough to show that $(r^{-e})^*\mathcal{G}$ is canonically isomorphic to \mathcal{G} as group schemes over C_R . Let us remark that the same construction gives an action of \mathbb{G}_m on $\widetilde{\mathcal{L}\mathcal{G}}$ (resp. $\widetilde{\mathcal{L}^+\mathcal{G}}$), compatible the dilatations on \tilde{C} . Furthermore, the action of $\widetilde{\mathcal{L}\mathcal{G}}$ (resp. $\widetilde{\mathcal{L}^+\mathcal{G}}$) on $\widetilde{\text{Gr}}_{\mathcal{G}}$ is \mathbb{G}_m -equivariant.

Let us define the action of \mathbb{G}_m on $C = \text{Spec} k[v]$ via $(r, v) \mapsto r^e v$. Observe that $\mu_e \subset \mathbb{G}_m$ acts trivially on C via this action.

Lemma 5.4. *Given the action of \mathbb{G}_m on C as above, there is a natural action \mathbb{G}_m on \mathcal{G} , such that $\mathcal{G} \rightarrow C$ is \mathbb{G}_m -equivariant.*

Remark 5.2. However, the natural dilatation on C could not lift to \mathcal{G} .

Proof. Let us denote the facet in the building of \mathcal{G}_{F_0} determined by $\mathcal{G}_{\mathcal{O}_0}$ by σ . Then under the embedding of buildings $\mathcal{B}(G, F) \subset \mathcal{B}(H \otimes \tilde{F}, \tilde{F})$, σ gives a facet in $\mathcal{B}(H \otimes \tilde{F}, \tilde{F})$, still denoted by σ . This in turn determines a parahoric group scheme of $H \otimes \tilde{F}$ over $\mathcal{O}_{\tilde{F}}$, denoted by $\tilde{\mathcal{G}}_\sigma$. Let $\tilde{\mathcal{G}}$ be the group scheme over \tilde{C} , such that $\tilde{\mathcal{G}}|_{\tilde{C}} = H \times \tilde{C}$, and $\tilde{\mathcal{G}}_{\mathcal{O}_{\tilde{F}}} = \tilde{\mathcal{G}}_\sigma$ (we make the obvious identification $\tilde{F} = F_0$). Then \mathcal{G} is the neutral connected component of $(\text{Res}_{\tilde{C}/C} \tilde{\mathcal{G}})^\Gamma$. To prove the proposition, it is enough to prove that there is a natural \mathbb{G}_m action on $\tilde{\mathcal{G}}$, compatible with the rotation on \tilde{C} . In addition, this \mathbb{G}_m -action commute with the action of Γ on $\tilde{\mathcal{G}}$.

Let $m, p : \mathbb{G}_m \times \tilde{C} \rightarrow \tilde{C}$ be the action map and the projection map respectively. We need show that there is an isomorphism of group schemes $p^*\tilde{\mathcal{G}} \cong m^*\tilde{\mathcal{G}}$ over $\mathbb{G}_m \times \tilde{C}$, satisfying the usually compatibility conditions. Since \mathbb{G}_m naturally acts on $\tilde{\mathcal{G}}|_{\tilde{C}} = H \times \tilde{C}$ by rotating the second factor, there is a natural isomorphism

$$c : p^*\tilde{\mathcal{G}}|_{\mathbb{G}_m \times \tilde{C}} \cong m^*\tilde{\mathcal{G}}|_{\mathbb{G}_m \times \tilde{C}},$$

which commutes with the Γ -actions. We need to show that this uniquely extends to an isomorphism over $\mathbb{G}_m \times \tilde{C}$. Then it will automatically commute of the Γ -actions and satisfy the compatibility conditions. Indeed, the uniqueness is clear

since $p^*\tilde{\mathcal{G}}$ (resp. $m^*\tilde{\mathcal{G}}$) is flat over $\mathbb{G}_m \times \tilde{C}$, so that $\mathcal{O}_{p^*\tilde{\mathcal{G}}} \subset \mathcal{O}_{p^*\tilde{\mathcal{G}}}[u^{-1}]$ (resp. $\mathcal{O}_{m^*\tilde{\mathcal{G}}} \subset \mathcal{O}_{m^*\tilde{\mathcal{G}}}[u^{-1}]$). We need to prove that the map $c : \mathcal{O}_{m^*\tilde{\mathcal{G}}}[u^{-1}] \rightarrow \mathcal{O}_{p^*\tilde{\mathcal{G}}}[u^{-1}]$ indeed sends $\mathcal{O}_{m^*\tilde{\mathcal{G}}} \rightarrow \mathcal{O}_{p^*\tilde{\mathcal{G}}}$. But this can be checked over each closed point of \mathbb{G}_m . Therefore, we reduce to prove that for every $r \in \mathbb{G}_m(k)$, the isomorphism of $r^*\tilde{\mathcal{G}}|_{\tilde{C}} \cong \tilde{\mathcal{G}}|_{\tilde{C}}$ extends to an isomorphism $r^*\tilde{\mathcal{G}} \cong \tilde{\mathcal{G}}$. We can replace \tilde{C} by $\mathcal{O}_{\tilde{0}}$. By [BT2, Proposition 1.7.6], it is enough to prove that the isomorphism $r : \tilde{\mathcal{G}}(F_{\tilde{0}}) \rightarrow \tilde{\mathcal{G}}(F_{\tilde{0}})$ induces the isomorphism $\tilde{\mathcal{G}}(\mathcal{O}_{\tilde{0}}) \rightarrow \tilde{\mathcal{G}}(\mathcal{O}_{\tilde{0}})$. But this follows from $\mathcal{G}(\mathcal{O}_{\tilde{0}}) = \text{ev}^{-1}(P)$, where P is the parabolic subgroup of H determined by the facet σ , and $\text{ev} : H(\mathcal{O}_{\tilde{0}}) \rightarrow H$ is the natural evaluation map. \square

It remains to show that each $\overline{\text{Gr}}_{\mathcal{G}, \mu}$ is invariant under this \mathbb{G}_m -action. It is enough to show that the section $s_\mu : \tilde{C} \rightarrow \overline{\text{Gr}}_{\mathcal{G}}$ is invariant under this \mathbb{G}_m -action. Recall that $s_\mu : \tilde{C} \rightarrow \text{Gr}_{\mathcal{G}} \times_C \tilde{C} \cong \text{Gr}_{H \times \tilde{C}}$ is given by the T_H -bundle $\mathcal{O}_{\tilde{C}}(\mu\Delta)$ with its canonical trivialization away from Δ (see Lemma 3.6). From this moduli interpretation, it is clear that s_μ is \mathbb{G}_m -invariant.

By restriction to $(\overline{\text{Gr}}_{\mathcal{G}})_{\tilde{0}} \cong \mathcal{F}\ell^Y$, we obtain an action of \mathbb{G}_m on $\mathcal{F}\ell^Y$. As is shown in [PR3], the affine flag variety $\mathcal{F}\ell^Y$ coincides with the affine flag variety in the Kac-Moody setting. Under this identification, the above \mathbb{G}_m -action on $\mathcal{F}\ell^Y$ corresponds to the action of the extra one-dimensional torus (usually called the rotation torus) in the maximal torus of the affine Kac-Moody group. We do not make the statement precise. Instead, we mention

Lemma 5.5. *Each Schubert variety in $\mathcal{F}\ell^Y$ is invariant under this action of \mathbb{G}_m on $\mathcal{F}\ell^Y$.*

Proof. Since \mathbb{G}_m acts on \mathcal{G} , it acts on $L^+\mathcal{G}_{\mathcal{O}_0}$. Clearly, it also acts on $L^+\mathcal{T}_{\mathcal{O}_0}$, and therefore acts on the the normalizer $N_{G(F_0)}(\mathcal{T}(\mathcal{O}_0))$ of $\mathcal{T}(\mathcal{O}_0)$ in $\mathcal{G}(F_0)$. Since $N_{G(F_0)}(\mathcal{T}(\mathcal{O}_0))/\mathcal{T}(\mathcal{O}_0) \cong \tilde{W}$ is discrete, the induced \mathbb{G}_m -action fixes every element. The lemma follows. \square

6. PROOFS I: FROBENIUS SPLITTING OF GLOBAL SCHUBERT VARIETIES

In this section, we prove Theorem 3.10 assuming Theorem 3.9. We also deduce Theorem 1 from Theorem 2.

6.1. Factorization of affine Demazure modules. In this subsection, we show that Theorem 2 implies Theorem 1. This is essentially proved in [Z]. Nevertheless, we sketch the proofs here since it serves as the prototype for the following subsections.

Let H be a split Chevalley group over k such that H_{der} is almost simple, simply-connected. Let Gr_H be the affine Grassmannian of H and \mathcal{L}_b be the line bundle on $(\text{Gr}_H)_{\text{red}}$ (the reduced subscheme of Gr_H), which restricts to the ample generator of the Picard group of each of connected component (which is isomorphic to $\text{Gr}_{H_{\text{der}}}$). We have the following two assertions.

Lemma 6.1. *Let $\mu \in \mathbb{X}_\bullet(T_H)$ be a minuscule coweight, so that $\overline{\text{Gr}}_\mu \cong X(\mu) = H/P(\mu)$. Then the restriction of \mathcal{L}_b to $\overline{\text{Gr}}_\mu$ is $\mathcal{L}(\mu)$, the ample generator of the Picard group of $X(\mu)$.*

Proof. Let us use the following notation. For ν a dominant weight of $P(\mu)$, let $\mathcal{L}(\nu)$ be the line bundle on $H/P(\mu)$, such that $\Gamma(H/P(\mu), \mathcal{L}(\nu))^*$ is the Weyl module of H of highest weight ν .

First assume that $\text{char } k = 0$. Let us fix a normalized invariant form $(\cdot, \cdot)_{\text{norm}}$ on $\mathbb{X}_\bullet(T_H)$ so that the square of the length of short coroots is two. Note that this invariant form may not be unique if H is not semisimple. For a coweight $\mu \in \mathbb{X}_\bullet(T_H)$ of T_H , let μ^* be the image of μ under $\mathbb{X}_\bullet(T_H) \rightarrow \mathbb{X}^\bullet(T_H)$ induced by this form. In other words, $(\mu^*, \lambda) = (\mu, \lambda)_{\text{norm}}$. Let $t_\mu \in \text{Gr}_{T_H}(k) \subset \text{Gr}_H(k)$ be the corresponding point as in the proof of Lemma 3.6. Now assume that μ is dominant. According to [Z, Lemma 2.2.2], the restriction of \mathcal{L}_b to $Ht_\mu \cong H/P(\mu)$ is isomorphic to $\mathcal{L}(\mu^*)$ (although their H -equivariant structure may be different). Therefore, the lemma follows from the fact that under the normalized invariant form, the square of the length of any minuscule coweight is two.

To prove the lemma in the case $\text{char } k > 0$, observe that everything is defined over \mathbb{Z} (see [Fa] where it is proved that the Schubert varieties are defined over \mathbb{Z} and commute with base change). It is well-known that $\text{Pic}(H/P(\mu)_{\mathbb{Z}}) \cong \text{Pic}(H/P(\mu)_k) \cong \mathbb{Z}$. The lemma follows. \square

Proposition 6.2. *Let \mathcal{L} be a line bundle on $(\text{Gr}_H)_{\text{red}}$, whose restriction to each connected component of Gr_H has the same central charge. Then*

$$H^0(\overline{\text{Gr}}_{\mu+\lambda}, \mathcal{L}) \cong H^0(\overline{\text{Gr}}_\mu, \mathcal{L}) \otimes H^0(\overline{\text{Gr}}_\lambda, \mathcal{L}).$$

Proof. Since $H^1(\overline{\text{Gr}}_\mu, \mathcal{L}) = 0$, it is enough to prove the proposition for $\mathcal{L}^n, \mathcal{L}^{2n}, \dots$ and some $n \geq 1$. Therefore we can replace \mathcal{L} by \mathcal{L}^n , we can assume that the central charge of \mathcal{L} is $2h^\vee$, i.e. $\mathcal{L} = \mathcal{L}_b^{2h^\vee}$. Then \mathcal{L}^n is the pullback of the n -tensor of the determinant line bundle $\mathcal{L}_{\text{det}}^n$ of $\text{Gr}_{\text{GL}(\text{Lie } H)}$ along $i : \text{Gr}_H \rightarrow \text{Gr}_{\text{GL}(\text{Lie } H)}$, as has been discussed in the proof of Theorem 4.3. Let us choose a complete curve (e.g. $\bar{C} = C \cup \{\infty\}$) and let Bun_H be the moduli stack of H -bundles on the curve. Then we know that \mathcal{L} is the pullback of a line bundle on Bun_H (in fact the anti-canonical bundle) along $\text{Gr}_H \rightarrow \text{Bun}_H$. Denote this line bundle on Bun_H as ω^{-1} . Of course, if H is simply-connected and semisimple, it is well-known that every line bundle on Gr_H is the pullback of some line bundle on Bun_H .

Consider the convolution affine Grassmannian $\text{Gr}_{H \times C}^{\text{Conv}}$ over C , defined as

$$\text{Gr}_{H \times C}^{\text{Conv}}(R) = \left\{ (y, \mathcal{E}, \mathcal{E}', \beta, \beta') \left| \begin{array}{l} y \in C(R), \mathcal{E}, \mathcal{E}' \text{ are two } H\text{-torsors on } C_R, \\ \beta : \mathcal{E}|_{C_R - \Gamma_y} \cong \mathcal{E}^0|_{C_R - \Gamma_y} \text{ is a trivialization,} \\ \text{and } \beta' : \mathcal{E}'|_{(C - \{0\})_R} \cong \mathcal{E}|_{(C - \{0\})_R}. \end{array} \right. \right\}.$$

This is an ind-scheme formally smooth over C , and we have

$$\text{Gr}_{H \times C}^{\text{Conv}}|_{\check{C}} \cong \text{Gr}_{H \times \check{C}} \times \text{Gr}_H, \quad (\text{Gr}_{H \times C}^{\text{Conv}})_0 \cong \text{Gr}_H \tilde{\times} \text{Gr}_H,$$

where $\text{Gr}_H \tilde{\times} \text{Gr}_H := LH \times^{L^+H} \text{Gr}_H$ is the local convolution Grassmannian. In addition, $\text{Gr}_{H \times C}^{\text{Conv}}$ is a fibration over $\text{Gr}_{H \times C}$ by sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to (y, \mathcal{E}, β) , with the fibers isomorphic to Gr_H .

Now $\overline{\text{Gr}}_\mu \times \overline{\text{Gr}}_\lambda$ extends naturally to a closed variety of $\text{Gr}_{H \times \check{C}} \times \text{Gr}_H$. The closure of this variety in $\text{Gr}_{H \times C}^{\text{Conv}}$ is denoted as $\overline{\text{Gr}}_{H \times C, \mu, \lambda}^{\text{Conv}}$. As is proven in [Z], for $y \neq 0$, $(\overline{\text{Gr}}_{H \times C, \mu, \lambda}^{\text{Conv}})_y \cong \overline{\text{Gr}}_\mu \times \overline{\text{Gr}}_\lambda$ and $(\overline{\text{Gr}}_{H \times C, \mu, \lambda}^{\text{Conv}})_0 \cong \overline{\text{Gr}}_\mu \tilde{\times} \overline{\text{Gr}}_\lambda$, where $\overline{\text{Gr}}_\mu \tilde{\times} \overline{\text{Gr}}_\lambda$ is the twisted product of $\overline{\text{Gr}}_\mu$ and $\overline{\text{Gr}}_\lambda$ (see *loc. cit.* or (6.2.6) below for the precise definition).

Let $h : \text{Gr}_{H \times C}^{\text{Conv}} \rightarrow \text{Bun}_H$ be the map sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to \mathcal{E}' . Then as explained in [Z], $h^*(\omega^{-1})^n$, when restricted to $\text{Gr}_{H \times C}^{\text{Conv}}|_{\check{C}}$, is isomorphic to $\mathcal{L}^n \boxtimes \mathcal{L}^n$, whereas over $(\text{Gr}_{H \times C}^{\text{Conv}})_0$, it is isomorphic to $m^* \mathcal{L}^n$, where $m : \text{Gr}_H \tilde{\times} \text{Gr}_H \rightarrow \text{Gr}_H$ is

the natural multiplication map. Therefore,

$$H^0(\overline{\mathrm{Gr}}_\mu, \mathcal{L}^n) \otimes H^0(\overline{\mathrm{Gr}}_\lambda, \mathcal{L}^n) \cong H^0(\overline{\mathrm{Gr}}_\mu \tilde{\times} \overline{\mathrm{Gr}}_\lambda, m^* \mathcal{L}^n) \cong H^0(\overline{\mathrm{Gr}}_{\mu+\lambda}, \mathcal{L}^n).$$

The last isomorphism is due to the fact $\mathcal{O}_{\overline{\mathrm{Gr}}_{\mu+\lambda}} \cong m_* \mathcal{O}_{\overline{\mathrm{Gr}}_\mu \tilde{\times} \overline{\mathrm{Gr}}_\lambda}$. \square

Clearly, these two assertions together with Theorem 2 will imply Theorem 1.

6.2. Reduction of Theorem 3.10 to Theorem 6.10. In this subsection, we prove Theorem 3.10, assuming Theorem 3.9. The key ingredient is the Frobenius splitting of varieties in characteristic p . We will assume that $\mathrm{char} k = p > 0$. In addition,

Lemma 6.3. *If Theorem 2 for one prime $p \nmid e$, then it holds for all $p \nmid e$ as well as the case $\mathrm{char} k = 0$.*

Proof. Recall that the affine flag varieties and Schubert varieties are defined $W(k)$, the ring of Witt vectors of k , and the formation commutes with base change ([Fa, PR3]). In addition, line bundles are also defined over $W(k)$ (for example, after identifying the affine flag varieties with those arising from Kac-Moody theory, this is clear. In fact, they are even defined over \mathbb{Z}' , where \mathbb{Z} is obtained from \mathbb{Z} by adding e th roots of unit and inverting e .) By the vanishing of corresponding H^1 , both sides are free $W(k)$ -modules and the formation commutes with base change. The lemma is clear. \square

So we just need to prove Theorem 3.10 for one prime. Therefore, we will assume $\mathrm{char} k = p > 2$. Let us remark that the only place we make use of this assumption is in the proof of Proposition 6.20. The assumption is not essential, but it will considerably simplify the proof.

We begin with introducing more varieties. Let $\mathrm{Gr}_{\mathcal{G}}^{BD}$ be the Beilinson-Drinfeld affine Grassmannian for \mathcal{G} over C . That is, for every k -algebra R ,

$$(6.2.1) \quad \mathrm{Gr}_{\mathcal{G}}^{BD}(R) = \left\{ (y, \mathcal{E}, \beta) \mid \begin{array}{l} y \in C(R), \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } C_R, \text{ and} \\ \beta : \mathcal{E}|_{(C-\{0\})_R - \Gamma_y} \cong \mathcal{E}^0|_{(C-\{0\})_R} \text{ is a trivialization} \end{array} \right\}.$$

Again, this is formally smooth over C , and we have

$$\mathrm{Gr}_{\mathcal{G}}^{BD}|_{\check{C}} \cong \mathrm{Gr}_{\mathcal{G}}|_{\check{C}} \times (\mathrm{Gr}_{\mathcal{G}})_0, \quad (\mathrm{Gr}_{\mathcal{G}}^{BD})_0 \cong (\mathrm{Gr}_{\mathcal{G}})_0 \cong \mathcal{F}\ell^Y.$$

We also need the convolution affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}^{Conv}$. The functor it represents is as follows. Let R be a k -algebra.

$$(6.2.2) \quad \mathrm{Gr}_{\mathcal{G}}^{Conv}(R) = \left\{ (y, \mathcal{E}, \mathcal{E}', \beta, \beta') \mid \begin{array}{l} y \in C(R), \mathcal{E}, \mathcal{E}' \text{ are two } \mathcal{G}\text{-torsors on } C_R, \\ \beta : \mathcal{E}|_{C_R - \Gamma_y} \cong \mathcal{G}|_{C_R - \Gamma_y} \text{ is a trivialization,} \\ \text{and } \beta' : \mathcal{E}'|_{(C-\{0\})_R} \cong \mathcal{E}|_{(C-\{0\})_R} \end{array} \right\}.$$

There is a natural map

$$(6.2.3) \quad m : \mathrm{Gr}_{\mathcal{G}}^{Conv} \rightarrow \mathrm{Gr}_{\mathcal{G}}^{BD}$$

sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to $(y, \mathcal{E}', \beta \circ \beta')$. This is a morphism over C , which is an isomorphism over $C - \{0\}$. Over 0, this morphism is the local convolution morphism

$$(6.2.4) \quad m : \mathcal{F}\ell^Y \tilde{\times} \mathcal{F}\ell^Y := LG \times^{L^+ \mathcal{G}_{F_Y}} \mathcal{F}\ell^Y \rightarrow \mathcal{F}\ell^Y,$$

given by the natural multiplication. There is another morphism

$$\pi : \mathrm{Gr}_{\mathcal{G}}^{Conv} \rightarrow \mathrm{Gr}_{\mathcal{G}}$$

sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to (y, \mathcal{E}, β) . This morphism makes $\mathrm{Gr}_{\mathcal{G}}^{Conv}$ a fibration (étale locally trivial) over $\mathrm{Gr}_{\mathcal{G}}$ with fibers isomorphic to $\mathcal{F}\ell^Y$. In addition, there is a section

$$z : \mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Gr}_{\mathcal{G}}^{Conv}$$

given by sending (y, \mathcal{E}, β) to $(y, \mathcal{E}, \mathcal{E}, \beta, \mathrm{id})$. Therefore, via z (resp. mz), $\mathrm{Gr}_{\mathcal{G}}$ is realized as closed subschemes of $\mathrm{Gr}_{\mathcal{G}}^{Conv}$ (resp. $\mathrm{Gr}_{\mathcal{G}}^{BD}$). We refer to [G] for the detailed discussions of the above facts.

Let $w \in W^Y \setminus \widetilde{W} / W^Y$ be an element in the extended affine Weyl group and let $\mathcal{F}\ell_w^Y$ denotes the corresponding Schubert variety in $\mathcal{F}\ell^Y$. Then $\overline{\mathrm{Gr}}_{\mu} \times \mathcal{F}\ell_w^Y \subset \mathrm{Gr}_H \times \mathcal{F}\ell^Y$ extends to a variety

$$\overline{\mathrm{Gr}}_{\mathcal{G}, \mu}|_{\check{C}} \times \mathcal{F}\ell_w^Y \subset (\mathrm{Gr}_{\mathcal{G}} \times_C \check{C}) \times \mathcal{F}\ell^Y \cong \mathrm{Gr}_{\mathcal{G}}^{BD} \times_C \check{C} \cong \mathrm{Gr}_{\mathcal{G}}^{Conv} \times_C \check{C}.$$

Let $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{BD}$ denote the flat closure of it in $\mathrm{Gr}_{\mathcal{G}}^{BD} \times_C \check{C}$, and $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{Conv}$ denote the flat closure of it in $\mathrm{Gr}_{\mathcal{G}}^{Conv} \times_C \check{C}$. Then $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{Conv}$ maps to $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{BD}$ via m . In addition, we have

Lemma 6.4. $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{Conv}$ is a fibration (via π) over $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu}$ with fibers isomorphic to $\mathcal{F}\ell_w^Y$.

Proof. Let us give another construction of $\mathrm{Gr}_{\mathcal{G}}^{Conv}$. There is a $L^+ \mathcal{G}_{\mathcal{O}_0}$ -torsor $\mathrm{Gr}_{\mathcal{G}, \underline{0}}$ over $\mathrm{Gr}_{\mathcal{G}}$ whose R -points classify

$$(6.2.5) \quad \mathrm{Gr}_{\mathcal{G}, \underline{0}}(R) = \left\{ (y, \mathcal{E}, \beta, \gamma) \left| \begin{array}{l} (y, \mathcal{E}, \beta) \in \mathrm{Gr}_{\mathcal{G}}(R), \text{ and a trivialization} \\ \gamma : \mathcal{E}|_{\{0\} \times \widehat{\mathrm{Spec} R}} \cong \mathcal{E}^0|_{\{0\} \times \widehat{\mathrm{Spec} R}} \end{array} \right. \right\}$$

where $\{0\} \times \widehat{\mathrm{Spec} R}$ is the completion of C_R along $\{0\} \times \mathrm{Spec} R$, either regarded as a formal scheme or the spectrum of its coordinate ring. Then

$$\mathrm{Gr}_{\mathcal{G}}^{Conv} \cong \mathrm{Gr}_{\mathcal{G}, \underline{0}} \times^{L^+ \mathcal{G}_{\mathcal{O}_0}} \mathcal{F}\ell^Y.$$

From this construction of $\mathrm{Gr}_{\mathcal{G}}^{Conv}$, it is clear that

$$\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{Conv} \cong \overline{\mathrm{Gr}}_{\mathcal{G}, \mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G}, \underline{0}} \times^{L^+ \mathcal{G}_{\mathcal{O}_0}} \mathcal{F}\ell_w^Y.$$

□

Proposition 6.5. Let $\nu \in \mathbb{X}_{\bullet}(T)$ be a sufficiently dominant coweight. Then the variety $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{BD}$ is normal, with the fiber $(\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{BD})_{\tilde{0}}$ over $\tilde{0} \in \check{C}$ being reduced.

Proof. The key observation is

Lemma 6.6. If ν is sufficiently large. Then the fiber $(\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{BD})_{\tilde{0}}$ is irreducible and generically reduced.

Assuming the lemma, then the proposition follows from Hironaka's lemma (cf. EGA IV.5.12.8). Namely, let V denote the underlying reduced subscheme of $(\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{BD})_{\tilde{0}}$. Then V is a Schubert variety of $\mathcal{F}\ell^Y$, which is normal by Theorem 2.3. Therefore, the proposition follows.

So it remains to prove the lemma. Let us first prove that $(\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{BD})_{\tilde{0}}$ is irreducible. Clearly, $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{Conv}$ maps surjectively onto $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{BD}$. Therefore, $(\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \nu}^{Conv})_{\tilde{0}}$ dominates

$(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$. We know that $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv})_{\tilde{0}}$ is a fibration over $(\overline{\text{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$, with fibers isomorphic to $\mathcal{F}\ell_\nu^Y$. Therefore by Theorem 3.9, the underlying reduced subschemes of irreducible components of $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv})_{\tilde{0}}$ are just

$$\mathcal{F}\ell_\lambda^Y \tilde{\times} \mathcal{F}\ell_\nu^Y, \quad \lambda \in \Lambda.$$

Here and after we use the following notation: let S_1, S_2 are two subschemes of $\mathcal{F}\ell^Y$ and assume that S_2 is $L^+\mathcal{G}_{\sigma_Y}$ stable, then we denote

$$(6.2.6) \quad S_1 \tilde{\times} S_2 := \widetilde{S_1} \times^{L^+\mathcal{G}_{\sigma_Y}} S_2,$$

where $\widetilde{S_1}$ is the preimage of S_1 under $L\mathcal{G}_{F_Y} \rightarrow \mathcal{F}\ell^Y$.

Therefore, the underlying reduced subscheme of each irreducible component of

$$(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}} \subset \mathcal{F}\ell^Y$$

is contained in one of $m(\mathcal{F}\ell_\lambda^Y \tilde{\times} \mathcal{F}\ell_\nu^Y), \lambda \in \Lambda$. Observe that when $\lambda \in \Lambda$ is not dominant, we have $\ell(t_{\nu+\lambda}) < \ell(t_\nu) + \ell(t_\lambda) = \ell(t_\nu) + \ell(t_\mu)$ for ν large enough by Lemma 2.1. However, by the flatness, all the irreducible components of $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$ have dimension $\ell(t_\mu) + \ell(t_\nu)$. This implies that $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$ has only one irreducible component, whose underlying reduced subscheme is $m(\mathcal{F}\ell_\mu^Y \tilde{\times} \mathcal{F}\ell_\nu^Y) = \mathcal{F}\ell_{\mu+\nu}^Y$.

Next, we prove that $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$ is generically reduced. In §5.2, we will construct the affine chart $c_\mu : U_\mu \subset \overline{\text{Gr}}_{\mathcal{G},\mu}$ satisfying:

- (1) $s_\mu(\tilde{C}) \subset U_\mu$;
- (2) U_μ is an affine space over \tilde{C} and therefore smooth over \tilde{C} ;
- (3) $(U_\mu)_{\tilde{0}} = C(\mu) \subset \mathcal{F}\ell^Y$ is the Schubert cell containing t_μ , i.e. the B -orbit containing t_μ .

Let us restrict $\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv}$ over U_μ . Then clearly, the fiber over $\tilde{0}$ of this family is $(U_\mu)_{\tilde{0}} \tilde{\times} \mathcal{F}\ell_\nu$ and therefore is irreducible and reduced. Let ξ be the generic point of $(U_\mu)_{\tilde{0}} \tilde{\times} \mathcal{F}\ell_\nu$. By the above argument, $\eta = m(\xi)$ is the generic point of $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$. Let A denote the local ring of $\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ at η and B denote the local ring of $\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv}$ at ξ . Both are discrete valuation rings, flat over \tilde{C} , and there is an injective map $A \rightarrow B$ which is an isomorphism over \tilde{C} . Since $\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv}$ is proper over \tilde{C} , we obtain a morphism $\text{Spec} A \rightarrow \overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv}$ which must factors through $\text{Spec} A \rightarrow \text{Spec} B$. That is, $A \rightarrow B$ is split injective. Therefore $A/uA \subset B/uB$ is a subfield. That is $\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ is generically reduced. \square

In fact, we proved that the fiber $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$ is isomorphic to $\mathcal{F}\ell_{\mu+\nu}^Y$.

If $\nu \in \mathbb{X}_\bullet(T_{\text{sc}})_\Gamma \subset \mathbb{X}_\bullet(T)_\Gamma$ so that $\nu \in W_{\text{aff}}$, then $z(\overline{\text{Gr}}_{\mathcal{G},\mu}) \subset \overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv}$ (resp. $mz(\overline{\text{Gr}}_{\mathcal{G},\mu}) \subset \overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$) is naturally a closed subscheme.

Theorem 6.7. *Assume that $\nu \in \mathbb{X}_\bullet(T_{\text{sc}})$ is a coweight dominant enough so that the above proposition holds. Then $\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ is Frobenius split, compatibly with $\overline{\text{Gr}}_{\mathcal{G},\mu}$ and $(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$.*

Corollary 6.8. *Theorem 3.10 holds. That is, the scheme $(\overline{\text{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ is reduced.*

Proof. This is because that

$$(\overline{\text{Gr}}_{\mathcal{G},\mu})_{\tilde{0}} = \overline{\text{Gr}}_{\mathcal{G},\mu} \cap (\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}},$$

and therefore is Frobenius split. In particular, it is reduced. \square

The remaining goal of this subsection is to reduce Theorem 6.7 to Theorem 6.10 via Proposition 6.9. Theorem 6.10 itself will be proven in later subsections. First, by standard argument it is enough to prove the theorem for the case $\mathcal{G}_{\mathcal{O}_0} \cong \mathcal{G}_C$ is the Iwahori group scheme. Therefore, from now on we assume that it is the case and write $B = L^+ \mathcal{G}_C$.

Since $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ is normal, we just need to find an open subscheme of $U \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$, whose complement has codimension at least two, such that U is Frobenius split, compatibly with $U \cap (\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$ and $U \cap \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. Therefore, we can throw away some bad loci of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ which is hard to control. In particular, we can throw away $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}} \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ which is of our main interests!

More precisely, we have

Proposition 6.9. *There is some open subscheme U of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{Conv}$, such that*

- (1) $m : \overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{Conv} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ maps U isomorphically onto an open subscheme $m(U)$ of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$, and the complement of $p(U)$ in $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ has codimension two;
- (2) U is Frobenius split, compatible with $U \cap (\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{Conv})_{\tilde{0}}$ and $U \cap z(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})$.

It is clear that Theorem 6.7 will follow from this proposition.

Proof. Let us first construct this open subscheme U . Recall that we constructed the section $s_\mu : \tilde{C} \rightarrow \widetilde{\mathrm{Gr}}_{\mathcal{G}}$ and $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is minimal irreducible closed subvariety of $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ that is invariant under $\mathcal{L}^+ \mathcal{G}$ and contains $s_\mu(\tilde{C})$. Let $\mathrm{Gr}_{\mathcal{G},\mu}$ denote the $\mathcal{L}^+ \mathcal{G}$ -orbit through s_μ . Then $\mathrm{Gr}_{\mathcal{G},\mu}$ is an open subscheme of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, which is smooth over \tilde{C} . In fact it is clear that $\mathrm{Gr}_{\mathcal{G},\mu}$ is open in $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ since the latter is the closure of the former. Therefore $\mathrm{Gr}_{\mathcal{G},\mu}$ is flat over \tilde{C} . Observe that under the isomorphism $\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C} \cong \mathrm{Gr}_H \times \tilde{C}$,

$$\mathrm{Gr}_{\mathcal{G},\mu}|_{\tilde{C}} \cong \mathrm{Gr}_\mu \times \tilde{C},$$

where Gr_μ denote the $L^+ H$ -orbit in Gr_H through t_μ , which is smooth. On the other hand $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}} = (U_\mu)_{\tilde{0}}$ is the Schubert cell $C(\mu)$ in $\mathcal{F}\ell$ containing μ , which is irreducible and smooth. Therefore, $\mathrm{Gr}_{\mathcal{G},\mu}$ is smooth over \tilde{C} .

Let U_1 be the preimage of $\mathrm{Gr}_{\mathcal{G},\mu}$ under $\pi : \overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{Conv} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. Then U_1 is a fibration over $\mathrm{Gr}_{\mathcal{G},\mu}$ with fibers $\mathcal{F}\ell_\nu$. As a scheme over \tilde{C} , the fiber of U_1 over $\tilde{0}$ is

$$C(\mu) \tilde{\times} \mathcal{F}\ell_\nu.$$

We define U to be the open subscheme of U_1 which coincides with U_1 over \tilde{C} , and which is given by

$$C(\mu) \tilde{\times} C(\nu) \subset C(\mu) \tilde{\times} \mathcal{F}\ell_\nu$$

over $\tilde{0}$.

We claim that $m : U \rightarrow m(U)$ is an isomorphism and the complement of $m(U)$ in $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ has codimension two. Over \tilde{C} , m is an isomorphism. Over $\tilde{0}$, the morphism

$$m : U_{\tilde{0}} \rightarrow (\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}}$$

is the same as

$$m : C(\mu) \tilde{\times} C(\nu) \rightarrow \mathcal{F}\ell_{\mu+\nu}.$$

It is well-known from the theory of Bott-Samelson-Demazure-Hasen (BSDH) resolution that m induces an isomorphism from $C(\mu) \tilde{\times} C(\nu)$ onto $C(\mu + \nu)$. Therefore, we have $m : U \rightarrow m(U)$ is a homeomorphism. Again, from the theory of BSDH resolution, we know that $m^{-1}m(U) = U$. Therefore, $m : U \rightarrow m(U)$ is a proper, birational homeomorphism with $m(U)$ normal, which must be an isomorphism. Notably, $(\overline{\text{Gr}}_{\mathcal{G},\mu})_{\tilde{0}} \subset \overline{\text{Gr}}_{\mathcal{G},\mu} \subset \overline{\text{Gr}}_{\mathcal{G}}^{BD,\lambda,\nu}$ is not contained in $m(U)$!

To see that the complement of $m(U)$ has codimension two, first observe that over $\overset{\circ}{C}$,

$$\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD}|_{\overset{\circ}{C}} - m(U)|_{\overset{\circ}{C}} \cong (\overline{\text{Gr}}_{\mu} - \text{Gr}_{\mu}) \times \mathcal{F}\ell_{\nu} \times \overset{\circ}{C},$$

which has codimension two, since $\overline{\text{Gr}}_{\mu} - \text{Gr}_{\mu}$ has codimension two in $\overline{\text{Gr}}_{\mu}$. Over $\tilde{0}$,

$$(\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD})_{\tilde{0}} - m(U)_{\tilde{0}} \cong \mathcal{F}\ell_{\mu+\nu} - C(\mu + \nu),$$

which has codimension at least one. This proves that the complement of $m(U)$ in $\overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{BD}$ has codimension two.

Next we turn to the second part of the proposition. Recall that U_1 is the preimage of $\text{Gr}_{\mathcal{G},\mu}$ under $\pi : \overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv} \rightarrow \overline{\text{Gr}}_{\mathcal{G},\mu}$. From the construction of U , we know that $U \subset U_1 \subset \overline{\text{Gr}}_{\mathcal{G},\mu,\nu}^{Conv}$. Therefore, it is enough to show that the same statement of Proposition 6.9 (2) holds for U_1 . Recall that

$$U_1 \cong (\text{Gr}_{\mathcal{G},\mu} \times_{\text{Gr}_{\mathcal{G}}} \text{Gr}_{\mathcal{G},\underline{0}}) \times^B \mathcal{F}\ell_{\nu},$$

where $\text{Gr}_{\mathcal{G},\underline{0}}$ is the B -torsor over $\text{Gr}_{\mathcal{G}}$ as constructed in the proof of Lemma 6.4. To simplify the notation, it is natural to denote

$$\text{Gr}_{\mathcal{G},\mu} \tilde{\times} V := \text{Gr}_{\mathcal{G},\mu} \times_{\text{Gr}_{\mathcal{G}}} \text{Gr}_{\mathcal{G},\underline{0}} \times^B V$$

for any B -variety V . Now, let $* \in \mathcal{F}\ell_{\nu}$ be the base point (recall that $\nu \in \mathbb{X}^{\bullet}(T_{\text{sc}})$, so that $*$, the Schubert variety corresponding to the identity element in the affine Weyl group, is contained in $\mathcal{F}\ell_{\nu}$). Then the closed embedding $z : \text{Gr}_{\mathcal{G},\mu} \rightarrow U_1$ corresponds to

$$\text{Gr}_{\mathcal{G},\mu} \tilde{\times} * \rightarrow \text{Gr}_{\mathcal{G},\mu} \tilde{\times} \mathcal{F}\ell_{\nu}.$$

Now the assertion follows from the following more general statement. □

Theorem 6.10. *For any $w \in \widetilde{W}$, there is a Frobenius splitting of $\text{Gr}_{\mathcal{G},\mu} \tilde{\times} \mathcal{F}\ell_w$, compatible with*

$$(\text{Gr}_{\mathcal{G},\mu} \tilde{\times} \mathcal{F}\ell_w)_{\tilde{0}} \cong (\text{Gr}_{\mathcal{G},\mu})_{\tilde{0}} \tilde{\times} \mathcal{F}\ell_w \cong C(\mu) \tilde{\times} \mathcal{F}\ell_w.$$

In addition, for any $v \leq w$ in \widetilde{W} , $\text{Gr}_{\mathcal{G},\mu} \tilde{\times} \mathcal{F}\ell_v \subset \text{Gr}_{\mathcal{G},\mu} \tilde{\times} \mathcal{F}\ell_w$ is also compatible with this splitting.

To remaining goal of this section is to prove this theorem.

6.3. Special parahorics. Let \mathcal{G}^s be the group scheme over C such that $\mathcal{G}_{\mathcal{O}_0}^s$ is a special parahoric group scheme of G . In this case, we can easily deduce Theorem 3.10 (assuming Theorem 3.9) directly from the Hironaka's lemma (without going into the argument presented in the previous subsection). This will in turn help us prove a special case of Theorem 6.10, namely, the case when $w = 1$ (see Corollary 6.18). Let us remark that if G is split, Proposition 6.17 is trivial and those who are only interested in split groups can go to the paragraph after this proposition directly.

So let $v \in A(G, S)$ be a special point in the apartment associated to (G, S) , and let \mathcal{G}_v be the corresponding special parahoric group scheme over \mathcal{O} . Let $\mathcal{F}\ell_v =$

$LG/L^+\mathcal{G}_v$ be the partial affine flag variety. To emphasize that it is the affine flag variety associated to a special parahoric, we sometimes also denote it by \mathcal{F}^s . As before, for each $\mu \in \mathbb{X}_\bullet(T)_\Gamma$, let us use t_μ to denote its lifting to $T(F)$ under the Kottwitz homomorphism $T(F) \rightarrow \mathbb{X}_\bullet(T)_\Gamma$. It gives a point in \mathcal{F}^s , still denoted by t_μ . Then the Schubert variety \mathcal{F}_μ^s is the closure of the $L^+\mathcal{G}_v$ -orbit in \mathcal{F}_v passing through t_μ . We have the following results special for Schubert varieties in \mathcal{F}^s , which generalizes the corresponding results for Gr_H (see also [Ri]).

Lemma 6.11. *The Schubert varieties are parameterized by $\mathbb{X}_\bullet(T)_\Gamma^+$. For $\mu \in \mathbb{X}_\bullet(T)_\Gamma^+$, the dimension of \mathcal{F}_μ^s is $(\mu, 2\rho)$. Let $\mathring{\mathcal{F}}_\mu^s \subset \mathcal{F}_\mu^s$ be the unique open $L^+\mathcal{G}_v$ -orbit in \mathcal{F}_μ^s . Then $\mathcal{F}_\mu^s - \mathring{\mathcal{F}}_\mu^s$ has codimension at least two.*

Proof. Observe that the natural map $\mathbb{X}_\bullet(T)_\Gamma^+ \subset \mathbb{X}_\bullet(T)_\Gamma \rightarrow W_0 \backslash \widetilde{W}/W_0$ is a bijection. The first claim follows. Let $B \subset L^+\mathcal{G}_v$ be an Iwahori subgroup of LG . Then the B -orbits in \mathcal{F}^s are parameterized by minimal length representatives in \widetilde{W}/W_0 . Let $\lambda \in \Lambda = W_0\mu \subset \mathbb{X}_\bullet(T)_\Gamma$. By Lemma 2.1, if $w \in \widetilde{W}$ is a minimal length representative for the coset $t_\lambda W_0$, then

$$\dim BwL^+\mathcal{G}_v/L^+\mathcal{G}_v \leq (\mu, 2\rho)$$

and if $\lambda \in \mathbb{X}_\bullet(T)_\Gamma^+$, the equality holds. Therefore, $\dim \mathcal{F}_\mu^s = (\mu, 2\rho)$. To prove the last claim, observe that if $\mathcal{F}_\lambda^s \subsetneq \mathcal{F}_\mu^s$, then $\mu - \lambda \in \mathbb{X}_\bullet(T_{\text{sc}})_\Gamma$ and therefore $(\mu - \lambda, 2\rho)$ is an even integer. \square

Recall that in [BD, §4.6], Beilinson and Drinfeld proved that $\overline{\text{Gr}}_\mu$ is Gorenstein, i.e., the dualizing sheaf $\omega_{\overline{\text{Gr}}_\mu}$ is indeed a line bundle. It is natural to ask whether the same results hold for \mathcal{F}_μ^s . However, the situation is more complicated in the ramified case dual to the fact that not all special points in the building of G are conjugate under $G_{\text{ad}}(F)$. More precisely, if $G_{\text{der}} \cong \text{SU}_{2n+1}$ is the odd ramified unitary group (see §8 for the definition), then there are two types of special parahoric group schemes (see Remark 8.1 (ii)).

Let us begin with the following lemma. Let v be any point in the apartment $A(G, S)$ and let \mathcal{G}_v be the corresponding parahoric group scheme for G . For simplicity, we write $K = L^+\mathcal{G}_v$. Then K acts on $\text{Lie}G$ by adjoint representation. Let $\mu \in \mathbb{X}_\bullet(T)_\Gamma$. Let

$$P = K \cap \text{Ad}_{t_\mu} K.$$

Then $\text{Lie}K$ and $\text{Ad}_{t_\mu} \text{Lie}K$ are P -modules.

Lemma 6.12. *As P -modules,*

$$(6.3.1) \quad \det \frac{\text{Lie}K}{\text{Lie}K \cap \text{Ad}_{t_\mu} \text{Lie}K} \cong (\det \frac{\text{Ad}_{t_\mu} \text{Lie}K}{\text{Lie}K \cap \text{Ad}_{t_\mu} \text{Lie}K})^{-1}.$$

Proof. Recall that we denote S to be the chosen maximal split F -torus of G . Its Néron model \mathcal{S} maps naturally to \mathcal{G}_v since $v \in A(G, S)$, and $L^+\mathcal{S}$ maps to P . Clearly, $\mathbb{X}^\bullet(P) \subset \mathbb{X}^\bullet(L^+\mathcal{S}) = \mathbb{X}^\bullet(S_k)$ (where S_k is the special fiber of \mathcal{S}). Therefore, it is enough to prove (6.3.1) as S_k -modules.

In §5.1.2, in particular Remark 5.1 (see also [PR3, 9.a, 9.b]), we will attach to each affine root α of (G, S) a 1-dimensional unipotent subgroup $U_\alpha \cong \mathbb{G}_a \subset LG$. They can be regarded as the "root subgroups" associated to α in the Kac-Moody setting. Let \mathfrak{u}_α be the Lie algebra. By definition

$$\text{Lie}K = \text{Lie}\mathcal{T}^{b,0} \oplus \prod_{\alpha(v) \geq 0} \mathfrak{u}_\alpha,$$

where $\mathcal{T}^{b,0}$ is the connected Néron model of T . Then clearly, as S_k -modules (we fix an embedding $S_k \rightarrow L^+\mathcal{S}$)

$$\frac{\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}K} \cong \bigoplus_{\alpha(v) \geq 0, \alpha(v-\mu) < 0} \mathfrak{u}_\alpha, \quad \frac{\mathrm{Ad}_{t_\mu}\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}K} \cong \bigoplus_{\alpha(v) < 0, \alpha(v-\mu) \geq 0} \mathfrak{u}_\alpha.$$

By identifying $A(G, S)$ with $\mathbb{X}_\bullet(S)_\mathbb{R}$ using the point v , we can write affine roots of G by $\alpha = a + m$, $a \in \Phi(G, S)$, where a is the vector part of α and $m = v(\alpha)$. Therefore, the set

$$\{\alpha(v) \geq 0, \alpha(v - \mu) < 0\} = \{a + m | a \in \Phi(G, S)^+, 0 \leq m < (\mu, a)\}$$

and

$$\{\alpha(v) < 0, \alpha(v - \lambda) \geq 0\} = \{a + m | a \in \Phi(G, S)^-, (\mu, a) \leq m < 0\}.$$

Since S_k acts on \mathfrak{u}_{a+m} via the weight a , the lemma is clear. \square

Remark 6.1. If G is split and v is a hyperspecial vertex, under some mild restriction of the characteristic of k , one can even show that as P -modules,

$$(6.3.2) \quad \frac{\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}K} \cong \left(\frac{\mathrm{Ad}_{t_\mu}\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}K} \right)^*.$$

This is because that there is an invariant non-degenerate symmetric bilinear form on $\mathrm{Lie}LG$ which induces a P -invariant non-degenerate bilinear form

$$\frac{\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}K} \otimes \frac{\mathrm{Ad}_{t_\mu}\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}K} \rightarrow k.$$

I do not know whether (6.3.2) is true if G is ramified and if v is a special vertex, but the argument fails.

Now we should specify the special vertex. If $G_{\mathrm{der}} \neq \mathrm{SU}_{2n+1}$, we can choose arbitrary special vertex in the building of G since they are conjugate under $G_{\mathrm{ad}}(F)$. If $G_{\mathrm{der}} = \mathrm{SU}_{2n+1}$, we choose the special vertex so that the corresponding parahoric group has reductive quotient Sp_{2n} (see Remark §8.1).

Theorem 6.13. *With the choice of the special vertex v as above, the Schubert variety $\mathcal{F}\ell_\mu^s$ is Gorenstein.*

Proof. As above, we denote \mathcal{G}_v to be the parahoric group of G corresponding to v and $K = L^+\mathcal{G}_v$. We need to show that the dualizing sheaf $\omega_{\mathcal{F}\ell_\mu^s}$ is indeed a line bundle. Let $\mathring{\mathcal{F}}\ell_\mu^s$ be the open K -orbit in $\mathcal{F}\ell_\mu^s$. Then we have shown that $\mathcal{F}\ell_\mu^s - \mathring{\mathcal{F}}\ell_\mu^s$ has codimension at least two. Let \mathcal{L}_{2c} be the pullback to $\mathcal{F}\ell^s$ of the determinant line bundle $\mathcal{L}_{\mathrm{det}}$ of $\mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\mathcal{G}_v)}$ along $i : \mathcal{F}\ell^s \rightarrow \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\mathcal{G}_v)}$. We first prove that $\omega_{\mathcal{F}\ell_\mu^s}^{-2} \cong \mathcal{L}_{2c}|_{\mathring{\mathcal{F}}\ell_\mu^s}$.

Indeed, observe that both sheaves are K -equivariant. The K -equivariant structure of $\omega_{\mathcal{F}\ell_\mu^s}^{-2}$ is induced from the action of K on $\mathring{\mathcal{F}}\ell_\mu^s$. On the other hand, a central extension of LG acts on \mathcal{L}_{2c} , and a splitting of this central extension over K defines a K -equivariant structure on \mathcal{L}_{2c} . To fix this K -equivariant structure uniquely, we will require that action of K on the fiber of \mathcal{L}_{2c} over $* \in \mathcal{F}\ell^s$ is trivial. Then the K -equivariant structure on \mathcal{L}_{2c} is given as follows (for simplicity, we only describe it at the level of k -points, but the generalization to R -points is clear, for example see cf. [FZ]): recall that for $x \in \mathcal{F}\ell^s$, $i(x)$ is a lattice in $\mathrm{Lie}G$ and $\mathcal{L}_{2c}|_x$ is the k -line

$$\mathcal{L}_{2c}|_x = \det(i(x)|\mathrm{Lie}K) := \det \frac{\mathrm{Lie}K}{\mathrm{Lie}K \cap i(x)} \otimes \det \left(\frac{i(x)}{\mathrm{Lie}K \cap i(x)} \right)^{-1}$$

Then for $g \in K$, $\mathcal{L}_{2c}|_x \rightarrow \mathcal{L}_{2c}|_{gx}$ is given by

$$\det(g) : \det(i(x)|\mathrm{Lie}K) \cong \det(i(gx)|g\mathrm{Lie}K) = \det(i(gx)|\mathrm{Lie}K).$$

Now it is enough to prove that there is an isomorphism $\mathcal{L}_{2c}|_{t_\mu} \cong \omega_{\mathcal{F}\ell_\mu^s}^{-2}|_{t_\mu}$ as 1-dimensional representations of $P = t_\mu K t_\mu^{-1} \cap K$, the stabilizer of $t_\mu \in \mathcal{F}\ell^s$ in K . Clearly,

$$\omega_{\mathcal{F}\ell_\mu^s}^{-2}|_{t_\mu} \cong (\det \frac{\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu} \mathrm{Lie}K})^2$$

as P -modules. On the other hand, it follows from the construction of the determinant line bundle that

$$\mathcal{L}_{2c}|_{t_\mu} \cong \det \frac{\mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu} \mathrm{Lie}K} \otimes (\det \frac{\mathrm{Ad}_{t_\mu} \mathrm{Lie}K}{\mathrm{Lie}K \cap \mathrm{Ad}_{t_\mu} \mathrm{Lie}K})^{-1}$$

as P -modules. Therefore, the assertion follows from Lemma 6.12.

Next, we prove that there is a line bundle \mathcal{L}_c on $(\mathcal{F}\ell^s)_{red}$ such that $\mathcal{L}_c^2 \cong \mathcal{L}_{2c}$. Indeed, for any $g \in G(F)$ acting on $\mathcal{F}\ell^s$ by left translation, we have $g^* \mathcal{L}_{2c} \cong \mathcal{L}_{2c}$. Therefore, it is enough to construct \mathcal{L}_c in the neutral connected component of $(\mathcal{F}\ell^s)_{red}$, which is isomorphic to $\mathcal{F}\ell_{sc}^s$, the corresponding affine flag variety for G_{der} by [PR3, §6]. Since v is a special vertex, $\mathrm{Pic}(\mathcal{F}\ell_{sc}^s) \cong \mathbb{Z}\mathcal{L}(\epsilon_i)$, where $i \in \mathbf{S}$ is a special vertex in the local Dynkin diagram of G corresponding to v . By checking the Kac's book [Kac, §4, §6] Under our choice of v , $a_i^\vee = 1$, (for SU_{2n+1} , there is another special vertex $i' \in \mathbf{S}$ such that $a_{i'}^\vee = 2$, and the reductive quotient of the corresponding parahoric group is SO_{2n+1} , see the following remark and Remark 8.1). Therefore, the central charge of $\mathcal{L}(\epsilon_i)$ is 1, whereas the central charge of \mathcal{L}_{2c} is $2h^\vee$ by Lemma 2.2.1. Therefore, $\mathcal{L}_c = \mathcal{L}(h^\vee \epsilon_i)$.

Now since the Picard group of $\mathcal{F}\ell_\mu^s$ has no-torsion, we must have $\omega_{\mathcal{F}\ell_\mu^s}^{-1} \cong \mathcal{L}_c|_{\mathcal{F}\ell_\mu^s}$, which in turn implies that $\omega_{\mathcal{F}\ell_\mu^s}^{-1} \cong \mathcal{L}_c$. \square

Observe the above proof implies that no matter what special vertex we choose, $\omega_{\mathcal{F}\ell_\mu^s}^{-2}$ is always a line bundle. The following corollary is what we need in the sequel.

Corollary 6.14. *For any special vertex v of G , $H^1(\mathcal{F}\ell_\mu^s, \omega_{\mathcal{F}\ell_\mu^s}^n) = 0$ for all $n < 0$.*

Remark 6.2. In the case $G_{der} = \mathrm{SU}_{2n+1}$, if we take the special vertex to be v_0 , the one defined by the pinning (2.1.1) so that \mathcal{G}_{v_0} is of the form (2.1.2), then the reductive quotient is SO_{2n+1} and the corresponding $a_i^\vee = 2$. Since the dual Coxeter number of SL_{2n+1} is $2n+1$, this means that on the partial flag variety $\mathcal{F}\ell^s$ corresponding to this special vertex, \mathcal{L}_{2c} does NOT have a square root. Let $B = L^+ \mathcal{G}_C$ be the Iwahori group of G_{der} corresponding to the chosen alcove, $i \in \mathbf{S}$ given by v_0 . Let $\mathbb{P}_i^1 = P_i/B$ be the rational line in $\mathcal{F}\ell_{sc} = LG_{der}/B$ as constructed in §2.2. It projects to a rational curve in $\mathcal{F}\ell^s$ under $LG_{der}/L^+ \mathcal{G}_C \rightarrow LG_{der}/L^+ \mathcal{G}_{v_0}$ (an explicit description of this rational line is given in (8.0.1)). Then the restriction of \mathcal{L}_{2c} to this rational line has degree $2n+1$. Since this rational is contained in any Schubert variety $\mathcal{F}\ell_\mu^s$, this means that $\omega_{\mathcal{F}\ell_\mu^s}^{-1}$ is NOT a line bundle, i.e. $\mathcal{F}\ell_\mu^s$ is not Gorenstein.

Now we turn to the global Schubert varieties. Let $\mathcal{G}^s = ((\mathrm{Res}_{\tilde{C}/C}(H \times \tilde{C}))^\Gamma)^0$ be the Bruhat-Tits group scheme over C as constructed in §3.2. Therefore $\mathcal{G}_{O_0}^s \cong \mathcal{G}_{v_0}$ is the special parahoric group scheme for \mathcal{G}_{F_0} as in (2.1.2).

Proposition 6.15. *Assume Theorem 3.9. Then Theorem 3.10 holds for \mathcal{G}^s .*

Proof. By Theorem 3.9, the support of $(\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu})_{\tilde{0}}$ is a single Schubert variety. This is because, when $\mathcal{G}_{\mathcal{O}_0}^s = \mathcal{G}_\sigma$ is a special parahoric group scheme, $W^Y = W_0$ and $W_0 \backslash \mathrm{Adm}^Y(\lambda)/W_0$ consists of only one extremal element in the Bruhat order, namely t_μ under the projection $\mathrm{Adm}^Y(\mu) \rightarrow W_0 \backslash \mathrm{Adm}^Y(\mu)/W_0$. This proves that the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ is irreducible. On the other hand, we have the affine chart U_μ (see §5.2) of $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ and $(U_\mu)_{\tilde{0}}$ is open in $(\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu})_{\tilde{0}}$. Therefore, the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ is generically reduced. By Hironaka's lemma again, $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ is normal over \tilde{C} , with special fiber reduced, indeed isomorphic to $\mathcal{F}\ell_\mu^s$. \square

Corollary 6.16. *The global Schubert variety $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ is normal and Cohen-Macaulay.*

Proof. The normality follows from the Hironaka's lemma. Since $\mathcal{F}\ell_\mu^s$ is Cohen-Macaulay, the assertion follows. \square

We refer to §9.2 for a brief discussion of the Frobenius splitting.

Proposition 6.17. *The variety $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ is Frobenius split, compatible with $(\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu})_{\tilde{0}}$.*

Proof. For simplicity, let us denote $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ by X . Then $f : X \rightarrow \tilde{C}$ is flat which is fiberwise normal and Cohen-Macaulay (since each X_y is a Schubert variety). Let $\omega_{X/\tilde{C}}$ be the relative dualizing sheaf on X . We know that $f_*\omega_{X/\tilde{C}}^{1-p}$ is a vector bundle on \tilde{C} by Corollary 6.14.

By the construction of §5.3, the sheaf $f_*\omega_{X/\tilde{C}}^{1-p}$ is \mathbb{G}_m -equivariant, and therefore, we can choose a \mathbb{G}_m -equivariant isomorphism

$$(6.3.3) \quad f_*\omega_{X/\tilde{C}}^{1-p} \cong H^0(X_{\tilde{0}}, \omega_{X_{\tilde{0}}}^{1-p}) \otimes \mathcal{O}_{\tilde{C}}.$$

Let $\sigma \in H^0(X_{\tilde{0}}, \omega_{X_{\tilde{0}}}^{1-p})$ be a \mathbb{G}_m -invariant section which splits $X_{\tilde{0}}$ (i.e. σ is a splitting of the natural map $\mathcal{O}_{X_{\tilde{0}}} \rightarrow F_*\mathcal{O}_{X_{\tilde{0}}}$, when regarded as a morphism from $F_*\mathcal{O}_{X_{\tilde{0}}} \rightarrow \mathcal{O}_{X_{\tilde{0}}}$ via (9.2.1)). Such a section always exists, since if $\tau \in H^0(X_{\tilde{0}}, \omega_{X_{\tilde{0}}}^{1-p})$ is any splitting section, and decompose $\tau = \sum \tau_j$ according to the \mathbb{G}_m -weights, then τ_0 is also a splitting section. Let $\sigma \otimes 1$ be a section of $f_*\omega_{X/\tilde{C}}^{1-p}$ via the isomorphism (6.3.3). We claim that $\sigma \otimes 1$, regarded as a morphism $(F_{X/\tilde{C}})_*\mathcal{O}_X \rightarrow \mathcal{O}_{X(p)}$ via (9.2.3), will maps 1 to 1. In fact, $(\sigma \otimes 1)(1)$ is a \mathbb{G}_m -invariant non-zero function on $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ since its restriction to $X_{\tilde{0}}$ is non-zero by (9.2.7). Since all regular functions on $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ come from \tilde{C} , $(\sigma \otimes 1)(1)$ is a \mathbb{G}_m -invariant non-zero function on \tilde{C} , which must be a constant. But its restriction to $X_{\tilde{0}}$ is 1, the claim follows.

Now, let $(\sigma \otimes 1) \otimes (\frac{u}{du})^{p-1} \in f_*\omega_{X/\tilde{C}}^{1-p} \otimes \omega_{\tilde{C}}^{1-p} \cong f_*\omega_X^{1-p}$. By the formula (9.2.2) and the commutative diagram (9.2.6), the proposition follows. \square

Let \mathcal{G} be the group scheme with $\mathcal{G}_{\mathcal{O}_0} = \mathcal{G}_C$. Let $B = L^+\mathcal{G}_C$. Observe that the natural projection $\mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Gr}_{\mathcal{G}^s}$ induces an isomorphism from $\mathrm{Gr}_{\mathcal{G}, \mu}$ to its image in $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$. To see this, observe that $\mathrm{Gr}_{\mathcal{G}, \mu}$ is covered by U_μ and $\mathrm{Gr}_{\mathcal{G}, \mu}|_{\tilde{C}^\circ}$, both of which maps isomorphically to their images in $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$. We thus regard $\mathrm{Gr}_{\mathcal{G}, \mu}$ as an open subscheme of $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu}$ under this map. The boundary $\overline{\mathrm{Gr}}_{\mathcal{G}^s, \mu} - \mathrm{Gr}_{\mathcal{G}, \mu}$ has codimension at least two. Therefore, we have proven

Corollary 6.18. *$\mathrm{Gr}_{\mathcal{G}, \mu}$ is Frobenius split, compatible with $(\mathrm{Gr}_{\mathcal{G}, \mu})_{\tilde{0}}$.*

Corollary 6.19. *All global functions on $\mathrm{Gr}_{\mathcal{G}, \mu}$ come from \tilde{C} .*

6.4. Proof of Theorem 6.10. The goal of this subsection is to prove Theorem 6.10. Without loss of generality, we can assume that $w \in W_{\text{aff}}$. Let $s_i (i \in \mathbf{S})$ be the simple reflections (determined by the alcove C). Let us recall that for $\tilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_m})$ being a sequence of simple reflections corresponding to affine simple roots. The Bott-Samelson-Demazure-Hasen variety is defined as

$$D_{\tilde{w}} = P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_m} / B,$$

where P_i is the parahoric group corresponding to i (so that $P_i/B \cong \mathbb{P}^1$). This is a smooth variety which is iterated fibration by \mathbb{P}^1 . For any subset $\{j_1, \dots, j_n\} \subset \{1, \dots, m\}$, let $\tilde{v} = (s_{i_{j_1}}, \dots, s_{i_{j_n}})$ be the subsequence of \tilde{w} , let $H_{i_p}, p = 1, 2, \dots, m$ be defined as

$$H_{i_p} = \begin{cases} B & \text{if } p \notin (j_1, \dots, j_n) \\ P_{i_p} & \text{if } p \in (j_1, \dots, j_n) \end{cases}$$

Then there is a closed embedding $\sigma_{\tilde{v}, \tilde{w}} : D_{\tilde{v}} \rightarrow D_{\tilde{w}}$ given by

$$(6.4.1) \quad D_{\tilde{v}} = P_{j_1} \times^B P_{j_2} \times^B \dots \times^B P_{j_n} / B \cong H_{i_1} \times^B \dots \times^B H_{i_m} / B \\ \hookrightarrow P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_m} / B = D_{\tilde{w}}.$$

In particular, let $\tilde{w}[j]$ denote the subsequence of \tilde{w} obtained by deleting s_{i_j} . Then

$$\sigma_{\tilde{w}[j], \tilde{w}} : D_{\tilde{w}[j]} \hookrightarrow D_{\tilde{w}}$$

is a divisor. In this way, we obtained m divisors of $D_{\tilde{w}}$. If \tilde{v}_1, \tilde{v}_2 are two subsequences of \tilde{w} , then the scheme-theoretical intersection $D_{\tilde{v}_1} \cap D_{\tilde{v}_2}$ inside $D_{\tilde{w}}$ is $D_{\tilde{v}_1 \cap \tilde{v}_2}$.

For $w \in W_{\text{aff}}$, let $m = \ell(w)$ and let us fix a reduced expression of $w = s_{i_1} \dots s_{i_m}$ and let $\tilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_m})$. Let $D_{\tilde{w}}$ be the corresponding BSDH variety so that $D_{\tilde{w}}$ is smooth and $\pi_{\tilde{w}} : D_{\tilde{w}} \rightarrow \mathcal{F}\ell_w$ is birational. By twisting by the B -torsor $\text{Gr}_{\mathcal{G}, \mu} \times_{\text{Gr}_{\mathcal{G}}} \text{Gr}_{\mathcal{G}, \underline{0}}$, we have $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}} \rightarrow \text{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w$, still denoted by $\pi_{\tilde{w}}$.

By the standard argument, to prove Theorem 6.10, it is enough to prove that:

Proposition 6.20. $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}}$ is Frobenius split, compatible with all $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}[j]}$ and $(\text{Gr}_{\mathcal{G}, \mu})_{\tilde{0}} \tilde{\times} D_{\tilde{w}}$.

Let $\omega_{D_{\tilde{w}}}$ be the canonical sheaf of $D_{\tilde{w}}$. It is well-known that there is an isomorphism (for example, see [Go1, Proposition 3.19] for the SL_n case, [PR3, proof of Proposition 9.6], [Ma, Ch. 8, 18] for the general case)

$$(6.4.2) \quad \omega_{D_{\tilde{w}}}^{-1} \cong \mathcal{O}(\sum_{j=1}^m D_{\tilde{w}[j]}) \otimes \pi_{\tilde{w}}^* \mathcal{L}(\sum_{i \in \mathbf{S}} \epsilon_i),$$

where $\mathcal{L}(\sum_{i \in \mathbf{S}} \epsilon_i)$ is the line bundle on $\mathcal{F}\ell_{\text{sc}}$ as defined in §2.2 (Recall that since we assume that $w \in W_{\text{aff}}$, $\mathcal{F}\ell_w \subset \mathcal{F}\ell_{\text{sc}} = (\mathcal{F}\ell)_{\text{red}}^0$ by [PR3, §6]). We need a relative version of this isomorphism.

Let us denote \mathcal{L}_{2c} be the line bundle on $\mathcal{F}\ell$ which is the pullback of \mathcal{L}_{det} along $\mathcal{F}\ell \rightarrow \text{Gr}_{\text{GL}(\text{Lie } B)}$ as before. It is naturally B -equivariant and therefore gives rise to a line bundle on $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell$ by twisting. We still denote this line bundle by \mathcal{L}_{2c} . In addition, to simplify the notation, let us denote the projection $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}} \rightarrow \text{Gr}_{\mathcal{G}, \mu}$ by $f : X \rightarrow V$. Then by the same proof of (6.4.2) (i.e. induction on the length of w), we have

$$(6.4.3) \quad \omega_{X/V}^{-2} \cong \mathcal{O}(2 \sum_{j=1}^m \text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}[j]}) \otimes \pi_{\tilde{w}}^* \mathcal{L}_{2c}.$$

Remark 6.3. Observe in the above isomorphism we use $\omega_{X/V}^{-2}$ rather than $\omega_{X/V}^{-1}$. The reason is as follows. Clearly, $\omega_{D_{\tilde{w}}}^{-1}$ is B -equivariant. If $G = G_{\text{der}}$ is simply-connected, both lines $\mathcal{O}(\sum_{j=1}^m D_{\tilde{w}[j]})$ and $\mathcal{L}(\sum_{i \in \mathbf{S}} \epsilon_i)$ are also B -equivariant and (6.4.2) is naturally upgraded to a B -equivariant isomorphism so by twisting the B -torsor $\text{Gr}_{\mathcal{G}, \mu} \times_{\text{Gr}_{\mathcal{G}}} \text{Gr}_{\mathcal{G}, \underline{0}}$ we can obtain a relative version of (6.4.2) whose square gives rise to (6.4.3). However, if G is not simply-connected, the B -equivariant structure on $\mathcal{O}(\sum_{j=1}^m D_{\tilde{w}[j]})$ or $\mathcal{L}(\sum_{i \in \mathbf{S}} \epsilon_i)$ may not exist nor be unique. Therefore the natural relative version for (6.4.2) is (6.4.3).

We will prove later on the following lemma.

Lemma 6.21. *There is a section σ_0 of \mathcal{L}_{2c} whose divisor $\text{div}(\sigma_0) \subset \text{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w$ does not intersect $z(\text{Gr}_{\mathcal{G}, \mu}) = \text{Gr}_{\mathcal{G}, \mu} \tilde{\times} *$.*

Let us remark that the line bundle $\mathcal{L}(\sum_{i \in \mathbf{S}} \epsilon_i)$ is very ample on $\mathcal{F}\ell_w$, and therefore there exists a section of $\mathcal{L}(\sum_{i \in \mathbf{S}} \epsilon_i)$ that does not pass through $*$. However, \mathcal{L}_{2c} is twisted by the B -torsor $\text{Gr}_{\mathcal{G}, \mu} \times_{\text{Gr}_{\mathcal{G}}} \text{Gr}_{\mathcal{G}, \underline{0}}$, and it is not ample. Therefore, some detailed analysis of this line bundle is needed.

Let us first assume this lemma, and let σ be a section of $\omega_{X/V}^{-2}$ whose divisor is of the form

$$(6.4.4) \quad \text{div}(\sigma) = 2 \sum_{j=1}^m \text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}[j]} + \text{div}(\pi_{\tilde{w}}^* \sigma_0).$$

We claim that

Lemma 6.22. *A non-zero scalar multiple of the section $\sigma^{\frac{p-1}{2}} \in \omega_{X/V}^{1-p}$ (recall that we assume that $p > 2$), when regarded as a morphism from $(F_{X/V})_* \mathcal{O}_X \rightarrow \mathcal{O}_{X^{(p)}}$ via (9.2.3), will send 1 to 1.*

Proof. First by Corollary 6.19, $\Gamma(X^{(p)}, \mathcal{O}_{X^{(p)}}) = \Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}})$. Let

$$h = \sigma^{\frac{p-1}{2}}(1) \in \Gamma(X^{(p)}, \mathcal{O}_{X^{(p)}})$$

be the function as in the lemma. It is indeed a function coming from \tilde{C} . We claim that this function is non-where vanishing. Then it must be a constant function on \tilde{C} . To see h is non-where vanishing, let $x \in \text{Gr}_{\mathcal{G}, \mu}$ be a point, and it is enough to show the restriction of h to $(D_{\tilde{w}})_x := \text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}}|_x \cong D_{\tilde{w}}$ is not zero. This is because the restriction of σ to $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}}|_x$ gives a divisor of the form $2 \sum_{j=1}^m D_{\tilde{w}[j]} + D$ for some D which does not passing through $*$. Therefore, by (a slight variant of) [MR, Proposition 8], $\sigma^{\frac{p-1}{2}}|_{(D_{\tilde{w}})_x}$, when regarded as a morphism from $F_* \mathcal{O}_{D_{\tilde{w}}}$ to $\mathcal{O}_{(D_{\tilde{w}})_x}$ via (9.2.1), will send 1 to a non-zero constant function on $(D_{\tilde{w}})_x$. Therefore, by (9.2.7),

$$h|_{(D_{\tilde{w}})_x} = \sigma^{p-1}|_{(D_{\tilde{w}})_x}(1)$$

is a non-zero constant. This finishes the proof of the lemma. \square

Now let $\tau \in \omega_{\text{Gr}_{\mathcal{G}, \mu}}^{1-p}$ be a section which gives rise to a Frobenius splitting of $\text{Gr}_{\mathcal{G}, \mu}$, compatible with $(\text{Gr}_{\mathcal{G}, \mu})_{\tilde{0}}$ by Corollary 6.18. Consider $\sigma^{\frac{p-1}{2}} \otimes f^* \tau \in \omega_V^{-1}$. By (9.2.6), it gives a splitting of $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}}$, compatible with $(\text{Gr}_{\mathcal{G}, \mu})_{\tilde{0}} \tilde{\times} D_{\tilde{w}}$. Again, by (a slight variant of) [MR, Proposition 8], this splitting is also compatible with all $\text{Gr}_{\mathcal{G}, \mu} \tilde{\times} D_{\tilde{w}[j]}$. This finishes the proof of Theorem 6.10.

It remains to prove Lemma 6.21. Let us consider

$$V_w = \Gamma(\mathcal{F}\ell_w, \mathcal{L}_{2c}) \rightarrow \Gamma(*, \mathcal{L}_{2c}) = V_1 \cong k.$$

It is surjective. By twisting the B -torsor $\mathrm{Gr}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}}$, we obtain a surjective morphism of vector bundles $\mathcal{V}_w \rightarrow \mathcal{V}_1 \cong \mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}}$ over $\mathrm{Gr}_{\mathcal{G},\mu}$. Clearly, \mathcal{V}_w is $\pi_* \mathcal{L}_{2c}$. Then to prove Lemma 6.21 is equivalent to prove that there is a morphism $\mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \rightarrow \mathcal{V}_w$ (which determines the section σ_0 of \mathcal{L}_{2c}) such that the composition $\mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \rightarrow \mathcal{V}_w \rightarrow \mathcal{V}_1$ is an isomorphism.

To this goal, let us first observe that the B -torsor $\mathrm{Gr}_{\mathcal{G},\underline{0}} \times_C \mathring{C} \rightarrow \mathrm{Gr}_{\mathcal{G}} \times_C \mathring{C}$ has a canonical section. Namely, we associated an R -point (y, \mathcal{E}, β) of $\mathrm{Gr}_{\mathcal{G}} \times_C \mathring{C}$ an R -point $(y, \mathcal{E}, \beta, \gamma)$ of $\mathrm{Gr}_{\mathcal{G},\underline{0}} \times_C \mathring{C}$ as follows. Since the graph Γ_y of $y : \mathrm{Spec} R \rightarrow C$ does not intersect with $\{0\} \times \mathrm{Spec} R \subset C \times \mathrm{Spec} R$, we can define

$$\gamma : \mathcal{E}|_{\{0\} \times \widehat{\mathrm{Spec} R}} \rightarrow \mathcal{E}^0|_{\{0\} \times \widehat{\mathrm{Spec} R}}$$

as the restriction of $\beta : \mathcal{E}|_{C_R - \Gamma_y} \cong \mathcal{E}^0|_{C_R - \Gamma_y}$. By base change, we get a canonical section (a canonical trivialization) ψ of the B -torsor $W \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}} \rightarrow W$, where $W = \mathrm{Gr}_{\mathcal{G},\mu}|_{\mathring{C}} \cong \mathrm{Gr}_{\mu} \times \mathring{C}$. Therefore, $\mathrm{Gr}_{\mathcal{G},\mu} \tilde{\times} \mathcal{F}\ell_w|_W \cong W \times \mathcal{F}\ell_w$ canonically, and over W , we have

$$\begin{array}{ccc} V_w \otimes \mathcal{O}_W & \longrightarrow & V_1 \otimes \mathcal{O}_W \\ \cong \downarrow & & \cong \downarrow \\ \mathcal{V}_w|_W & \longrightarrow & \mathcal{V}_1|_W \end{array}$$

To complete the proof of the lemma, it is enough to show

- (1) the isomorphism $V_1 \otimes \mathcal{O}_W \rightarrow \mathcal{V}_1|_W$ extends to an isomorphism $V_1 \otimes \mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \rightarrow \mathcal{V}_1$;
- (2) there is a splitting $V_1 \rightarrow V_w$ (equivalently, a section of $\mathcal{L}(2 \sum_{i \in S} \epsilon_i)$ whose divisor does not pass through $* \in \mathcal{F}\ell_w$), such that the induced map

$$V_1 \otimes \mathcal{O}_W \rightarrow V_w \otimes \mathcal{O}_W \rightarrow \mathcal{V}_w|_W$$

extends to $V_1 \otimes \mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \rightarrow \mathcal{V}_w$.

Let us first prove (1). Recall that $\mathrm{Gr}_{\mathcal{G},\mu}$ is covered by W and U_μ , it is enough to prove that $V_1 \otimes \mathcal{O}_{U_\mu \cap W} \rightarrow \mathcal{V}_1|_{U_\mu \cap W}$ extends to an isomorphism $V_1 \otimes \mathcal{O}_{U_\mu} \rightarrow \mathcal{V}_1|_{U_\mu}$. Recall that U_μ is an affine space and therefore the B -torsor $U_\mu \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}} \rightarrow U_\mu$ is trivial (since B , or more precisely, every finite dimensional quotient of B has a filtration by \mathbb{G}_m s and \mathbb{G}_a s). Choose a trivialization α . Then over $U_\mu \cap W$, the two trivializations α and ψ differ by an element $g \in B(U_\mu \cap W)$. In general, let M be a (finite dimensional) B -module given by $\rho : B \rightarrow \mathrm{GL}(M)$, and let \mathcal{M} be the associated vector bundle. Then α induces $\mathcal{M} \cong M \otimes \mathcal{O}_{U_\mu}$ and ψ induces $\mathcal{M}|_W \cong M \otimes \mathcal{O}_W$, and these two trivializations differ by $\rho(g) \in \mathrm{GL}(M)(U_\mu \cap W)$. In our case, the B -module V_1 is a trivial module, therefore the claim follows.

To prove (2), we shall give another construction of \mathcal{L}_{2c} . Let us complete the curve $\bar{C} = C \cup \{\infty\} \cong \mathbb{P}^1$ and extend \mathcal{G} to a group scheme over \bar{C} as in §3.2 (so that $\mathcal{G}_{\mathcal{O}_\infty}$ is special). Let $\mathrm{Bun}_{\mathcal{G}}$ be the moduli stack of \mathcal{G} -bundles on \bar{C} . Let us express $\mathcal{F}\ell$ as the ind-scheme representing (\mathcal{E}, β) , where \mathcal{E} is a \mathcal{G} -torsor on \bar{C} and β a trivialization of \mathcal{E} away from $0 \in \bar{C}$. Then there is a natural morphism $h : \mathcal{F}\ell \rightarrow \mathrm{Bun}_{\mathcal{G}}$. Let $\omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}$ be the anti-canonical bundle of $\mathrm{Bun}_{\mathcal{G}}$. Then by definition the pullback of $\omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}$ to $\mathcal{F}\ell$ is \mathcal{L}_{2c} . Now, consider the following morphisms

$$\mathrm{Bun}_{\mathcal{G}} \xleftarrow{h_1} \mathrm{Gr}_{\mathcal{G}} \xleftarrow{\pi} \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}} \xrightarrow{m} \mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}} \xrightarrow{h_2} \mathrm{Bun}_{\mathcal{G}}.$$

Lemma 6.23. *Over $\mathrm{Gr}_{\mathcal{G}} \tilde{\times} \mathcal{F}\ell_w \subset \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$, there is an isomorphism*

$$\mathcal{L}_{2c} \cong m^* h_2^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1} \otimes \pi^* h_1^* \omega_{\mathrm{Bun}_{\mathcal{G}}}.$$

Proof. Since $\mathrm{Gr}_{\mathcal{G}} \tilde{\times} \mathcal{F}\ell_w$ is proper over $\mathrm{Gr}_{\mathcal{G}}$, by the see-saw principle, it is enough to show that: (i) for each $x \in \mathrm{Gr}_{\mathcal{G}}$, the restrictions of $m^* h_2^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1} \otimes \pi^* h_1^* \omega_{\mathrm{Bun}_{\mathcal{G}}}$ and \mathcal{L}_{2c} to $\mathcal{F}\ell_w \subset \pi^{-1}(x)$ are isomorphic; and (ii) when restricting both line bundles on $z : \mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$, they are isomorphic.

Indeed, recall that over \mathring{C} , $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}|_{\mathring{C}} \cong \mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}|_{\mathring{C}} \cong \mathrm{Gr}_{\mathcal{G}}|_{\mathring{C}} \times \mathcal{F}\ell$, and over $0 \in C(k)$, the morphisms $(\mathrm{Gr}_{\mathcal{G}})_0 \xleftarrow{\pi} (\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}})_0 \xrightarrow{m} (\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}})_0$ looks like $\mathcal{F}\ell \xleftarrow{\pi} \mathcal{F}\ell \tilde{\times} \mathcal{F}\ell \xrightarrow{m} \mathcal{F}\ell$. Under these isomorphisms

$$h_2^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}|_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}|_{\mathring{C}}} \cong h_1^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}|_{\mathrm{Gr}_{\mathcal{G}}|_{\mathring{C}}} \otimes h^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}, \quad h_2^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}|_{(\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}})_0} \cong h^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}.$$

Therefore, for all $x \in \mathrm{Gr}_{\mathcal{G}}$, the restriction of $m^* h_2^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1} \otimes \pi^* h_1^* \omega_{\mathrm{Bun}_{\mathcal{G}}}$ to $\mathcal{F}\ell_w \subset \pi^{-1}(x)$ is isomorphic to \mathcal{L}_{2c} . The first fact is established. For the second fact, one can easily see that when restricting both line bundles on $z : \mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$, they are isomorphic to the trivial bundle. \square

Another preparation we need for the proof of (2) is

Lemma 6.24. *There exists a section $\Theta \in \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}$ (over the neutral connected component of $\mathrm{Bun}_{\mathcal{G}}$), whose associated divisor does not pass through the trivial \mathcal{G} -torsor.*

Proof. Let Bun_n denote the moduli stack of rank n vector bundles on \mathbb{P}^1 . Let us review the theta divisor on the stack whose associated line bundle is the determinant line bundle $\mathcal{L}_{\mathrm{det}}$. Recall that Bun_n breaks into connected components labeled by d , the degree of the bundle. Also recall that each rank n bundle on \mathbb{P}^1 can be written as $\oplus_{i=1}^n \mathcal{O}(d_i)$. Then the deformation theory shows that if $|d_i - d_j| \leq 1$ for any $1 \leq i, j \leq n$, then the bundle $\oplus_{i=1}^n \mathcal{O}(d_i)$ forms an open substack $\mathrm{Bun}_n^{d, \circ} \subset \mathrm{Bun}_n^d$, where $d = \sum d_i$, and the complement is (by definition) the theta divisor on the this connected component.

Now let Θ be the pullback of the theta divisor on along $\mathrm{Bun}_{\mathcal{G}} \rightarrow \mathrm{Bun}_{\mathrm{GL}(\mathrm{Lie}_{\mathcal{G}})}$. We need to show that this divisor does not pass through the trivial \mathcal{G} -torsor. Observe that the Lie algebra of the group scheme $(\mathrm{Res}_{\bar{C}/\bar{C}}(H \times \bar{C}))^{\Gamma}$ is a trivial $\mathcal{O}_{\bar{C}}$ -module (indeed, it is $\mathcal{V} = ([e]_*(\mathrm{Lie} H \otimes \mathcal{O}_{\bar{C}}))^{\Gamma}$, which is a direct summand of $[e]_*(\mathrm{Lie} H \otimes \mathcal{O}_{\bar{C}})$). Since $\mathcal{V}(-0) \subset \mathcal{V}$, in the decomposition $\mathrm{Lie}_{\mathcal{G}} = \oplus \mathcal{O}(d_i)$, we have $|d_i - d_j| \leq 1$ and the lemma follows. \square

The pullback of Θ along $h : \mathcal{F}\ell \rightarrow \mathrm{Bun}_{\mathcal{G}}$ gives rise to a section σ^0 of \mathcal{L}_{2c} , whose associated divisor does not pass through the base point $* \in \mathcal{F}\ell_w$. This gives us a splitting $V_1 \rightarrow V_w$ which we claim is the desired splitting satisfying (2).

Indeed, since the B -torsor $\mathrm{Gr}_{\mathcal{G}, \underline{0}} \times_C \mathring{C} \rightarrow \mathrm{Gr}_{\mathcal{G}} \times_C \mathring{C}$ has a canonical section, we can spread out σ^0 as a section of \mathcal{L}_{2c} over $\mathrm{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w|_W$, still denoted by σ^0 . This corresponds the induces map $V_1 \otimes \mathcal{O}_W \rightarrow V_w \otimes \mathcal{O}_w$. Then to prove (2), it is equivalent to show that σ^0 indeed extends to a section of \mathcal{L}_{2c} over the whole $\mathrm{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w$. Otherwise, let $n > 0$ be the smallest integer such that $u^n \sigma^0$ would extend (recall that we use u to denote the global coordinate on \tilde{C} so that $u = 0$ defines the divisor $(\mathrm{Gr}_{\mathcal{G}, \mu})_{\tilde{0}} \tilde{\times} \mathcal{F}\ell_w$ inside $\mathrm{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w$). Then $u^n \sigma^0|_{(\mathrm{Gr}_{\mathcal{G}, \mu})_{\tilde{0}} \tilde{\times} \mathcal{F}\ell_w}$ would not be zero. Observe that by construction, over $\mathrm{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w|_W$, we have

$$\pi^* h_1^* \Theta \otimes \sigma^0 = m^* h_2^* \Theta,$$

as sections in $m^* h_2^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}|_{\mathrm{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w|_W}$. Then as sections in $m^* h_2^* \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}$ over the whole $\mathrm{Gr}_{\mathcal{G}, \mu} \tilde{\times} \mathcal{F}\ell_w$, we would have

$$\pi^* h_1^* \Theta \otimes u^n \sigma^0 = u^n m^* h_2^* \Theta.$$

When restricting this equation to $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}} \tilde{\times} \mathcal{F}\ell_w$, the right hand side is zero. However, the left hand side is not since $\pi^* h_1^* \Theta|_{(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}} \tilde{\times} \mathcal{F}\ell_w} \neq 0$. Contradiction!

7. PROOFS II: THE NEARBY CYCLES

7.1. Central sheaves: proof of Theorem 3.9. In this subsection, we prove Theorem 3.9. As mentioned in the introduction, a direct proof would be to write down a moduli problem \mathcal{M}_μ over \tilde{C} , which is a closed subscheme of $\widehat{\mathrm{Gr}}_{\mathcal{G}}$, such that: (i) $\mathcal{M}_\mu|_{\tilde{C}} \cong \overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}}$; and (ii) $(\mathcal{M}_\mu)_{\tilde{0}}(k) = \bigcup_{w \in \mathrm{Adm}^Y(\mu)} \mathcal{F}\ell_w^Y(k)$. Then by Lemma 3.8, Theorem 3.9 would follow. Unfortunately, so far, such a moduli functor is not available for general group G and general coweight μ . In certain cases, such a moduli problem is available. We refer to [PRS] for a survey of the known results.

The proof presented here is indirect. The observation is as follows. Let (S, s, η) be a Henselian trait, i.e. S is the spectrum of a discrete valuation ring, s is the closed point of S and η is the generic point of S . Assume that the residue field $k(s)$ of s is algebraically closed and let ℓ be a prime different from $\mathrm{char} k(s)$. For V a variety over a field whose characteristic prime to ℓ , the intersection cohomology sheaf is the Goresky-Macpherson extension to V of the (shifted) constant sheaf $\mathbb{Q}_\ell[\dim V]$ on the smooth locus of V .

Lemma 7.1. *Let $f : V \rightarrow S$ be a proper flat morphism. Let IC be the intersection cohomology sheaf of $V_\eta := V \times_S \eta$ and let $\Psi_V(\mathrm{IC})$ be the nearby cycle of IC . Then the support of $\Psi_V(\mathrm{IC})$ is V_s .*

Proof. Let $x \in V$ be a point in the special fiber V_s and \bar{x} be a geometric point over x . Then by definition $\Psi_V(\mathrm{IC})_{\bar{x}} \cong H^*((V_{(\bar{x})})_{\bar{\eta}}, \mathrm{IC}|_{(V_{(\bar{x})})_{\bar{\eta}}})$, where $V_{(\bar{x})}$ is the strict Henselization of V at \bar{x} , and $(V_{(\bar{x})})_{\bar{\eta}}$ is its fiber over $\bar{\eta}$, a geometric point over η . Let x be a generic point of V_s , then $(V_{(\bar{x})})_{\bar{\eta}}$ is a point and $\mathrm{IC}|_{(V_{(\bar{x})})_{\bar{\eta}}} \cong \mathbb{Q}_\ell[\dim V]$. The lemma is clear. \square

Now, let ℓ be a prime different from p . Let IC_μ be the intersection cohomology sheaf of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}}$, then the nearby cycle $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$ is a perverse on $\mathcal{F}\ell^Y$ whose support is $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$. Therefore, to prove the theorem, it is enough to determine the support of $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$. In fact, we will give a filtration of $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$ and describe the support of each associated graded piece.

When the group G is split, such a description can be deduced from [AB, Theorem 4] directly. We will mostly follow their strategy for the ramified case with the following difference. We will not make use of the results in Appendix to [Be] and therefore we will not generalize the full version of [AB, Theorem 4] to the ramified case (but see Remark 7.2). In particular, we will not perform any categorical arguments as in *loc. cit.*.

Let us denote $K^Y = L^+ \mathcal{G}_{\sigma_Y}$, and let $\mathrm{P}_{K^Y}(\mathcal{F}\ell^Y)$ denote the category of K^Y -equivariant perverse sheaves on $\mathcal{F}\ell^Y$. Recall that this category is defined as the direct limit of categories of K^Y -equivariant of perverse sheaves supported on the K^Y -stable finite dimensional subvarieties of $\mathcal{F}\ell^Y$.

Lemma 7.2. *The sheaf $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$ naturally belongs to $\mathrm{P}_{K^Y}(\mathcal{F}\ell^Y)$.*

Proof. Let $\mathcal{L}_n^+ \mathcal{G}$ be the n th jet group of \mathcal{G} , i.e. the group scheme over C , whose R -points classifying (y, β) where $y \in C(R)$ and $\beta \in \mathcal{G}(\Gamma_{y,n})$, where $\Gamma_{y,n}$ is the n th nilpotent thickening of Γ_y . It is clear that $\mathcal{L}_n^+ \mathcal{G}$ is smooth over C and the action of $\mathcal{L}_n^+ \mathcal{G}$ on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ factors through some $\mathcal{L}_n^+ \mathcal{G} \times_C \tilde{C}$ for n sufficiently large.

Let $m : \mathcal{L}_n^+ \mathcal{G} \times_{\tilde{C}} \overline{\text{Gr}}_{\mathcal{G}, \mu} \rightarrow \overline{\text{Gr}}_{\mathcal{G}, \mu}$ be the multiplication and p be the natural projection. Then there is a canonical isomorphism $m^* \text{IC}_\mu \cong p^* \text{IC}_\mu$ as sheaves on $\mathcal{L}_n^+ \mathcal{G} \times_{\tilde{C}} \overline{\text{Gr}}_{\mathcal{G}, \mu}|_{\tilde{C}^\circ}$. By taking the nearby cycles, we have a canonical isomorphism

$$\Psi_{\mathcal{L}_n^+ \mathcal{G} \times_{\tilde{C}} \overline{\text{Gr}}_{\mathcal{G}, \mu}}(m^* \text{IC}_\mu) \cong \Psi_{\mathcal{L}_n^+ \mathcal{G} \times_{\tilde{C}} \overline{\text{Gr}}_{\mathcal{G}, \mu}}(p^* \text{IC}_\mu).$$

Since both m and p are smooth morphisms and taking nearby cycle commutes with smooth base change, we have

$$m^* \Psi_{\overline{\text{Gr}}_{\mathcal{G}, \mu}}(\text{IC}_\mu) \cong p^* \Psi_{\overline{\text{Gr}}_{\mathcal{G}, \mu}}(\text{IC}_\mu).$$

The compatibility of this isomorphism under further pullback follows from the corresponding compatibility for $m^* \text{IC}_\mu \cong p^* \text{IC}_\mu$. The lemma follows. \square

Definition 7.1. Define $\mathcal{Z}_\mu = \Psi_{\overline{\text{Gr}}_{\mathcal{G}, \mu}}(\text{IC}_\mu)$.

Let $D(\mathcal{F}\ell^Y)$ be the derived category of constructible sheaves on $\mathcal{F}\ell^Y$ and $D_{K^Y}(\mathcal{F}\ell^Y)$ be the K^Y -equivariant derived category on $\mathcal{F}\ell^Y$. Recall that $D_{K^Y}(\mathcal{F}\ell^Y)$ is a monoidal category and there is a monoidal action (the "convolution product") of $D_{K^Y}(\mathcal{F}\ell)$ on $D(\mathcal{F}\ell^Y)$. Namely, we have the convolution diagram

$$\mathcal{F}\ell^Y \times \mathcal{F}\ell^Y \xleftarrow{q} LG \times \mathcal{F}\ell^Y \xrightarrow{p} LG \times^{K^Y} \mathcal{F}\ell^Y = \mathcal{F}\ell^Y \tilde{\times} \mathcal{F}\ell^Y \xrightarrow{m} \mathcal{F}\ell^Y$$

Let $\mathcal{F}_1 \in D(\mathcal{F}\ell^Y)$, $\mathcal{F}_2 \in D_{K^Y}(\mathcal{F}\ell^Y)$, and let $\mathcal{F}_1 \tilde{\times} \mathcal{F}_2$ be the unique sheaf on $LG \times^{K^Y} \mathcal{F}\ell^Y$ such that

$$(7.1.1) \quad p^*(\mathcal{F}_1 \tilde{\times} \mathcal{F}_2) \cong q^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

Then

$$(7.1.2) \quad \mathcal{F}_1 \star \mathcal{F}_2 = m_!(\mathcal{F}_1 \tilde{\times} \mathcal{F}_2).$$

In general, if $\mathcal{F}_1, \mathcal{F}_2$ are perverse sheaves, it is not necessarily the case that $\mathcal{F}_1 \star \mathcal{F}_2$ is perverse. However, we have

Theorem 7.3. (i) Let \mathcal{F} be arbitrary perverse sheaf on $\mathcal{F}\ell^Y$. Then $\mathcal{F} \star \mathcal{Z}_\mu$ is a perverse sheaf on $\mathcal{F}\ell^Y$.

(ii) If $\mathcal{F} \in D_{K^Y}(\mathcal{F}\ell^Y)$, then there is a canonical isomorphism $c_{\mathcal{F}} : \mathcal{F} \star \mathcal{Z}_\mu \cong \mathcal{Z}_\mu \star \mathcal{F}$.

Remark 7.1. (i) In the case when $G = H$ is a split group, this theorem is proved by Gaitsgory (cf. [G])⁶. The general case proved below follows his line. Still, we take the opportunity to spell out all the details for the following reasons. First, the family $\text{Gr}_{\mathcal{G}}$ we used here is in fact different from Gaitsgory's family which has no obvious generalization to the ramified groups. On the other hand, this theorem for ramified groups will be used in the future to establish the geometrical Satake correspondence for ramified groups. Secondly, the usage of the non-constant group schemes allows us to simplify the arguments of Gaitsgory's. Namely, we can treat (i) and (ii) in Proposition 7.4 below equally. And this argument will be generalized to mixed characteristic situation in [PZ]. However, in the original arguments of Gaitsgory, the proof of (i) of Proposition 7.4 is considerably harder than the proof of (ii).

(ii) To simplify the notation, in the proof we only consider $Y = C$ being an alcove. In this case, we denote $B = K^C$ to be the Iwahori subgroup of LG , and denote $\mathcal{F}\ell = \mathcal{F}\ell^C$. However, the proof (with the only change by replacing B by K^Y and $\mathcal{F}\ell$ by $\mathcal{F}\ell^Y$) is valid in *any* parahoric case.

⁶In fact, Part (ii) of the theorem was proved in [G] under the assumption that \mathcal{F} is perverse. I am not sure whether the argument applies to the case \mathcal{F} being a complex.

Proof. Recall the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}^{BD}$ as introduced in (6.2.1). We have

$$\mathrm{Gr}_{\mathcal{G}}^{BD} \times_C \mathring{\tilde{C}} \cong \mathcal{F}\ell \times (\mathrm{Gr}_{\mathcal{G}} \times_C \mathring{\tilde{C}}).$$

For $\mathcal{F} \in D(\mathcal{F}\ell)$, let

$$\mathcal{F} \boxtimes \mathrm{IC}_{\mu} \subset D(\mathcal{F}\ell \times (\mathrm{Gr}_{\mathcal{G}} \times_C \mathring{\tilde{C}}))$$

be the sheaf on $\mathrm{Gr}_{\mathcal{G}}^{BD} \times_C \mathring{\tilde{C}}$. Consider the nearby cycle functor $\Psi_{\mathrm{Gr}_{\mathcal{G}}^{BD} \times_C \mathring{\tilde{C}}}$.

Proposition 7.4. (i) *If $\mathcal{F} \in D(\mathcal{F}\ell)$, there is a canonical isomorphism*

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{BD} \times_C \mathring{\tilde{C}}}(\mathcal{F} \boxtimes \mathrm{IC}_{\mu}) \cong \mathcal{F} \star \mathcal{Z}_{\mu}.$$

(ii) *If $\mathcal{F} \in D_B(\mathcal{F}\ell)$, then there is a canonical isomorphism*

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{BD} \times_C \mathring{\tilde{C}}}(\mathcal{F} \boxtimes \mathrm{IC}_{\mu}) \cong \mathcal{Z}_{\mu} \star \mathcal{F}.$$

It is clear that this proposition will imply the theorem. So it remains to prove the proposition.

We first prove (ii). Let $\mathrm{Gr}_{\mathcal{G}}^{Conv}$ be the convolution Grassmannian as introduced in (6.2.2), which we recall is a fibration over $\mathrm{Gr}_{\mathcal{G}}$ with fibers isomorphic to $\mathcal{F}\ell$. Regard $\mathcal{F} \boxtimes \mathrm{IC}_{\mu}$ as a sheaf on $\mathrm{Gr}_{\mathcal{G}}^{Conv} \times_C \mathring{\tilde{C}} \cong \mathcal{F}\ell \times (\mathrm{Gr}_{\mathcal{G}} \times_C \mathring{\tilde{C}})$. Since taking nearby cycles commutes with proper push-forward, it is enough to prove that as sheaves on $\mathcal{F}\ell \tilde{\times} \mathcal{F}\ell$, there is a canonical isomorphism

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{Conv} \times_C \mathring{\tilde{C}}}(\mathcal{F} \boxtimes \mathrm{IC}_{\mu}) \cong \mathcal{Z}_{\mu} \tilde{\times} \mathcal{F},$$

where $\mathcal{Z}_{\mu} \tilde{\times} \mathcal{F}$ is the twisted product as defined in (7.1.1).

Recall the B -torsor $\mathrm{Gr}_{\mathcal{G}, \underline{0}}$ as defined in (6.2.5) and $\mathrm{Gr}_{\mathcal{G}}^{Conv} \cong \mathrm{Gr}_{\mathcal{G}, \underline{0}} \times^B \mathcal{F}\ell$. Let $V \subset \mathcal{F}\ell$ be the support of \mathcal{F} , and $B_n = L_n^+ \mathcal{G}_{\mathcal{O}_0}$ (the n th jet group as defined in the proof of Lemma 7.2) be the finite dimensional quotient of B such that the action of B on V factors through B_n . Let $\mathrm{Gr}_{\mathcal{G}, 0, n}$ be the B_n -torsor over $\mathrm{Gr}_{\mathcal{G}}$ classifies $(y, \mathcal{E}, \beta, \gamma)$ where (y, \mathcal{E}, β) is in the definition of $\mathrm{Gr}_{\mathcal{G}}$ and γ is a trivialization of \mathcal{E} around the n th infinitesimal neighborhood of $0 \in C$. Then $\mathrm{IC}_{\mu} \tilde{\times} \mathcal{F}$ is supported on

$$(\tilde{C} \times_C \mathrm{Gr}_{\mathcal{G}, \underline{0}}) \times^B V \cong (\tilde{C} \times_C \mathrm{Gr}_{\mathcal{G}, 0, n}) \times^{B_n} V \subset \mathrm{Gr}_{\mathcal{G}}^{Conv} \times_C \mathring{\tilde{C}}.$$

Observe that over $\mathring{\tilde{C}}$, it makes sense to talk about $\mathrm{IC}_{\mu} \tilde{\times} \mathcal{F}$ (as defined via (7.1.1)), which is canonically isomorphic to $\mathcal{F} \boxtimes \mathrm{IC}_{\mu}$, we thus need to show that

$$(7.1.3) \quad \Psi_{\mathrm{Gr}_{\mathcal{G}}^{Conv} \times_C \mathring{\tilde{C}}}(\mathrm{IC}_{\mu} \tilde{\times} \mathcal{F}) \cong \mathcal{Z}_{\mu} \tilde{\times} \mathcal{F}.$$

Let us denote the pullback of IC_{μ} to $\mathrm{Gr}_{\mathcal{G}, 0, n} \times_C \mathring{\tilde{C}}$ by $\widetilde{\mathrm{IC}}_{\mu}$. Since $\mathrm{Gr}_{\mathcal{G}, 0, n} \rightarrow \mathrm{Gr}_{\mathcal{G}}$ is smooth, $\Psi_{\mathrm{Gr}_{\mathcal{G}, 0, n} \times_C \mathring{\tilde{C}}}(\widetilde{\mathrm{IC}}_{\mu})$ is canonically isomorphic to the pullback of \mathcal{Z}_{μ} , and

$$\Psi_{(\mathrm{Gr}_{\mathcal{G}, 0, n} \times_C \mathring{\tilde{C}}) \times V}(\widetilde{\mathrm{IC}}_{\lambda} \boxtimes \mathcal{F}) \cong \Psi_{\mathrm{Gr}_{\mathcal{G}, 0, n} \times_C \mathring{\tilde{C}}}(\widetilde{\mathrm{IC}}_{\lambda}) \boxtimes \mathcal{F}$$

is B_n -equivariant. We thus have (7.1.3).

Next we prove (i). There is another convolution affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}^{Conv'}$, which represents the functor to associate every k -algebra R ,

$$(7.1.4) \quad \mathrm{Gr}_{\mathcal{G}}^{Conv'}(R) = \left\{ (y, \mathcal{E}, \mathcal{E}', \beta, \beta') \left| \begin{array}{l} y \in C(R), \mathcal{E}, \mathcal{E}' \text{ are two } \mathcal{G}\text{-torsors on } C_R, \\ \beta : \mathcal{E}|_{(C-\{0\})_R} \cong \mathcal{E}^0|_{(C-\{0\})_R} \text{ is a trivialization,} \\ \text{and } \beta' : \mathcal{E}'|_{C_R-\Gamma_y} \cong \mathcal{E}|_{C_R-\Gamma_y} \end{array} \right. \right\}.$$

Clearly, we have $m' : \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'} \rightarrow \mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}$ by sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to $(y, \mathcal{E}', \beta' \circ \beta)$. This is a morphism over C , which is an isomorphism over $C - \{0\}$, and m'_0 again is the local convolution diagram

$$m : \mathcal{F}\ell \tilde{\times} \mathcal{F}\ell \rightarrow \mathcal{F}\ell.$$

Again, regard $\mathcal{F} \boxtimes \mathrm{IC}_{\mu}$ as a sheaf on $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'}|_{\tilde{C}} \cong \mathcal{F}\ell \times (\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C})$, it is enough to prove that as sheaves on $\mathcal{F}\ell \tilde{\times} \mathcal{F}\ell$,

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'} \times_C \tilde{C}}(\mathcal{F} \boxtimes \mathrm{IC}_{\mu}) \cong \mathcal{F} \tilde{\times} \mathcal{Z}_{\mu}.$$

Let $\mathcal{L}_n^+ \mathcal{G}$ be the n th jet group of \mathcal{G} . As mentioned before, $\mathcal{L}_n^+ \mathcal{G}$ is smooth over C and the action of $\mathcal{L}_n^+ \mathcal{G}$ on $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu}$ factors through some $\mathcal{L}_n^+ \mathcal{G} \times_C \tilde{C}$ for n sufficiently large.

Let us define the $\mathcal{L}_n^+ \mathcal{G}$ -torsor \mathcal{P}_n over $\mathcal{F}\ell \times C$ as follows. Its R -points are quadruples $(y, \mathcal{E}, \beta, \gamma)$, where $y \in C(R)$, (\mathcal{E}, β) are as in the definition of $\mathcal{F}\ell$ (and therefore β is a trivialization of \mathcal{E} on \tilde{C}_R), and γ is a trivialization of \mathcal{E} over $\Gamma_{y, n}$, the n th nilpotent thickening of the graph Γ_y of y . Then we have the twisted product

$$(\mathcal{P}_n \times_C \tilde{C}) \times^{\mathcal{L}_n^+ \mathcal{G} \times_C \tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G}, \mu} \subset \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'} \times_C \tilde{C}.$$

Over \tilde{C} , we can form the twisted product $\mathcal{F}[1] \tilde{\times} \mathrm{IC}_{\mu}$ as in (7.1.1), which is canonically isomorphic to $\mathcal{F} \boxtimes \mathrm{IC}_{\mu}$. By the same argument as in the proof of (ii) (i.e. by pulling back everything to $\mathcal{P}_n \times_{\tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G}, \mu}$), we have

$$\Psi_{(\mathcal{P}_n \times_C \tilde{C}) \times^{\mathcal{L}_n^+ \mathcal{G} \times_C \tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G}, \mu}}(\mathcal{F}[1] \tilde{\times} \mathrm{IC}_{\mu}) \cong \Psi_{\mathcal{F}\ell \times \tilde{C}}(\mathcal{F}[1]) \tilde{\times} \Psi_{\overline{\mathrm{Gr}}_{\mathcal{G}, \mu}}(\mathrm{IC}_{\mu}) \cong \mathcal{F} \tilde{\times} \mathcal{Z}_{\mu}.$$

□

Our goal to prove that the support of \mathcal{Z}_{μ} is exactly the Schubert varieties in $\mathcal{F}\ell^Y$ labeled by the set $W^Y \setminus \mathrm{Adm}^Y(\mu)/W^Y$, which will imply Theorem 3.9 by Lemma 7.1. Clearly, it is enough to prove this in the case $\mathcal{G}_{\mathcal{O}_0}$ is Iwahori.

Let us recall some standard objects in $\mathrm{P}_B(\mathcal{F}\ell)$. Recall that B -orbits in $\mathcal{F}\ell$ are labeled by elements $w \in \tilde{W}$. For any w , let $j_w : C(w) \rightarrow \mathcal{F}\ell_w$ be the open embedding of the Schubert cell to the Schubert variety. This is an affine embedding. Let us denote

$$j_{w*} = (j_w)_* \mathbb{Q}_{\ell}[\ell(w)], \quad j_{w!} = (j_w)_! \mathbb{Q}_{\ell}[\ell(w)].$$

Then it is well-known that there are canonical isomorphisms

$$(7.1.5) \quad \begin{aligned} j_{w*} \star j_{w'*} &\cong j_{ww'*}, & j_{w!} \star j_{w'!} &\cong j_{ww'!}, & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ j_{w*} \star j_{w^{-1}!} &\cong j_{w^{-1}!} \star j_{w*} && \cong \delta_e. \end{aligned}$$

In addition, if $\ell(ww'w'') = \ell(w) + \ell(w') + \ell(w'')$, then the two isomorphisms from $j_{w*} \star j_{w'*} \star j_{w''*}$ (resp. from $j_{w!} \star j_{w'!} \star j_{w''!}$) to $j_{ww'w''*}$ (resp. to $j_{ww'w''!}$) are the same.

Let us recall the following fundamental result due to I. Mirkovic (cf. [AB, Appendix]). The proof for ramified groups is exactly the same as for the split groups. In fact, the proof works in the general Kac-Moody setting.

Proposition 7.5. *Let $w, v \in \tilde{W}$, then both $j_{w*} \star j_{v!}$ and $j_{w!} \star j_{v*}$ are perverse sheaves. In addition, both sheaves are supported on the Schubert variety $\mathcal{F}\ell_{wv}$ and $j_{wv}^*(j_{w*} \star j_{v!}) \cong j_{wv}^*(j_{w!} \star j_{v*}) \cong \mathbb{Q}_{\ell}[\ell(wv)]$.*

Fix $w \in W_0$ to be an element in the finite Weyl group of G . We are going to define the w -Wakimoto sheaves on $\mathcal{F}\ell$. Recall the definition of $\mathbb{X}_\bullet(T)_\Gamma^+$ in (2.1.5). For $\mu \in \mathbb{X}_\bullet(T)_\Gamma$, we write $\mu = \lambda - \nu$ with $\lambda, \nu \in w(\mathbb{X}_\bullet(T)_\Gamma^+)$. Define

$$(7.1.6) \quad J_\mu^w = j_{t_\lambda}! \star j_{t_\nu}^*,$$

which is well-defined up to a canonical isomorphism (by (7.1.5)). By Proposition 7.5, $J_\mu^w \in P_B(\mathcal{F}\ell)$ and is supported on $\mathcal{F}\ell_\mu$ with $j_{t_\mu}^* J_\mu^w \cong \mathbb{Q}_\ell[\ell(t_\mu)]$. Let us remark that for G being split and $w = e$ being the identity element, they are the Wakimoto sheaves considered in [AB]. In addition, we have

$$(7.1.7) \quad J_\mu^w \star J_\lambda^w \cong J_{\mu+\lambda}^w.$$

In fact, by (7.1.5) and Lemma 2.1, this is true for μ, λ for $\mu, \lambda \in w(\mathbb{X}_\bullet(T)_\Gamma^+)$. The extension to all μ, λ is immediate.

One of the important applications of the Wakimoto sheaves is the following. Let $\mathcal{F} \in P_B(\mathcal{F}\ell)$. It is called convolution exact if $\mathcal{F}' \star \mathcal{F}$ is perverse for any $\mathcal{F}' \in P_B(\mathcal{F}\ell)$, and is called central if in addition $\mathcal{F} \star \mathcal{F}' \cong \mathcal{F}' \star \mathcal{F}$. For example, \mathcal{Z}_μ is central. The following proposition generalizes [AB, Proposition 5], where the case $w = e$ is considered. The proof is basically the same

Proposition 7.6. *Fix $w \in W_0$. Any central object in $P_B(\mathcal{F}\ell)$ has a filtration whose associated graded pieces are $J_\lambda^w, \lambda \in \mathbb{X}_\bullet(T)_\Gamma$.*

Proof. We begin with some general notations and remarks following [AB]. For a triangulated category D and a set of objects $S \subset Ob(D)$, let $\langle S \rangle$ be the set of all objects obtained from elements of S by extensions; i.e. $\langle S \rangle$ is the smallest subset of $Ob(D)$ containing $S \cup \{0\}$ and such that:

- (1) if $A \cong B$ and $A \in \langle S \rangle$, then $B \in \langle S \rangle$; and
- (2) for all $A, B \in \langle S \rangle$ and an exact triangle $A \rightarrow C \rightarrow B \rightarrow A[1]$, we have $C \in \langle S \rangle$.

Let $\mathcal{F} \in D_B(\mathcal{F}\ell)$. The $*$ -support of \mathcal{F} is defined to be

$$W_{\mathcal{F}}^* := \{w \in \widetilde{W} \mid j_w^* \mathcal{F} \neq 0\},$$

and the $!$ -support of \mathcal{F} is the set

$$W_{\mathcal{F}}^! := \{w \in \widetilde{W} \mid j_w^! \mathcal{F} \neq 0\}.$$

By the induction on the dimension of the support of \mathcal{F} , it is easy to see that if $\mathcal{F} \in D_B(\mathcal{F}\ell)^{p, \leq 0}$ (p stands for the perverse t-structure), then \mathcal{F} is contained in $\langle j_{v!}[n] \mid v \in W_{\mathcal{F}}^*, n \geq 0 \rangle$. On the other hand, if $\mathcal{F} \in D_B(\mathcal{F}\ell)^{p, \geq 0}$, then $\mathcal{F} \in \langle j_{v*}[n] \mid v \in W_{\mathcal{F}}^!, n \leq 0 \rangle$.

For any $\mathcal{F} \in D_B(\mathcal{F}\ell)$, there exists a finite subset $S_{\mathcal{F}} \subset \widetilde{W}$, such that

$$W_{j_{w*} \star \mathcal{F}}^!, W_{j_{w!} \star \mathcal{F}}^* \subset w \cdot S_{\mathcal{F}}; \quad W_{\mathcal{F} \star j_{w*}}^!, W_{\mathcal{F} \star j_{w!}}^* \subset S_{\mathcal{F}} \cdot w.$$

Namely, let $\mathcal{F}\ell_v$ be a Schubert variety such that \mathcal{F} is supported in $\mathcal{F}\ell_v$ (in both $*$ - and $!$ -sense). Then by the proper base change theorem, the above assertions will follow if we can show that there exists $S_v \subset \widetilde{W}$ such that

$$C(w) \tilde{\times} \mathcal{F}\ell_v \subset \bigcup_{v' \in w S_v} C(v'), \quad \mathcal{F}\ell_v \tilde{\times} C(w) \subset \bigcup_{v' \in S_v w} C(v'),$$

This can be proved easily by induction of the length of v .

Now we prove the proposition. Let $\mathcal{F} \in P_B(\mathcal{F}\ell)$ be a central object, and let $S_{\mathcal{F}} \subset \widetilde{W}$ be the finite set associated to \mathcal{F} as above. Recall that we have the special

vertex v_0 in the building of G , which determines an isomorphism $\widetilde{W} = \mathbb{X}_\bullet(T)_\Gamma \rtimes W_0$ determined by v_0 . Let $\mu \in w(\mathbb{X}_\bullet(T)_\Gamma^+)$ such that

$$t_\mu S_{\mathcal{F}} \subset w(\mathbb{X}_\bullet(T)_\Gamma^+) W_0, \quad S_{\mathcal{F}} t_\mu \subset W_0 w(\mathbb{X}_\bullet(T)_\Gamma^+).$$

This is always possible since $S_{\mathcal{F}}$ is a finite set. We have $J_\mu^w = j_{\mu!}$ and from $J_\mu^w \star \mathcal{F} \cong \mathcal{F} \star J_\mu^w$, we have

$$W_{J_\mu^w \star \mathcal{F}}^* \subset t_\mu S_{\mathcal{F}} \cap S_{\mathcal{F}} t_\mu \subset w(\mathbb{X}_\bullet(T)_\Gamma^+) W_0 \cap W_0 w(\mathbb{X}_\bullet(T)_\Gamma^+) = w(\mathbb{X}_\bullet(T)_\Gamma^+).$$

Therefore, $J_\mu^w \star \mathcal{F} \in \langle j_{t_\lambda!}[n] \mid \lambda \in w(\mathbb{X}_\bullet(T)_\Gamma^+), n \geq 0 \rangle$. Observe that $J_\lambda^w = j_{t_\lambda!}$ for $\lambda \in w(\mathbb{X}_\bullet(T)_\Gamma^+)$. Then by (7.1.7), we have

$$\mathcal{F} \in \langle J_\lambda^w[n] \mid \lambda \in \mathbb{X}_\bullet(T)_\Gamma, n \geq 0 \rangle.$$

By choosing $\mu \in w(-\mathbb{X}_\bullet(T)_\Gamma^+)$ large enough and using $J_\lambda^w = j_{t_\lambda*}$ for $\lambda \in w(-\mathbb{X}_\bullet(T)_\Gamma^+)$, we have

$$(7.1.8) \quad \mathcal{F}' := j_{t_\mu*} \star \mathcal{F} = J_\mu^w \star \mathcal{F} \in \langle j_{t_\lambda*}[n] \mid \lambda \in w(-\mathbb{X}_\bullet(T)_\Gamma^+), n \geq 0 \rangle.$$

We claim that this already implies that \mathcal{F}' has a filtration (in the category of perverse sheaves) with associated graded by $j_{t_\mu*}, \mu \in w(-\mathbb{X}_\bullet(T)_\Gamma^+)$, and therefore implies the proposition. Indeed, since \mathcal{F}' is perverse, for any $\nu \in w(-\mathbb{X}_\bullet(T)_\Gamma^+)$, the $!$ -stalk of \mathcal{F}' at t_ν has homological degree $\geq -\ell(t_\nu)$. On the other hand, (7.1.8) implies that the $!$ -stalk of \mathcal{F}' at t_ν has homological degree $\leq -\ell(t_\nu)$. The claim follows. \square

To proceed, we would like to some have study of the category of perverse sheaves on $\mathcal{F}\ell$ that are generated by those J_λ^w .

Lemma 7.7. *For $\lambda, \mu \in \mathbb{X}_\bullet(T)_\Gamma$, $R\mathrm{Hom}(J_\lambda^w, J_\mu^w) = 0$ unless $w^{-1}(\lambda) \preceq w^{-1}(\mu)$. Furthermore, $R\mathrm{Hom}(J_\mu^w, J_\mu^w) \cong \bar{\mathbb{Q}}_\ell$.*

Proof. $R\mathrm{Hom}(J_\lambda^w, J_\mu^w) = R\mathrm{Hom}(J_{\lambda+\nu}^w, J_{\mu+\nu}^w) = R\mathrm{Hom}(j_{t_{\lambda+\nu}!}, j_{t_{\mu+\nu}!})$ for $\nu \in \mathbb{X}_\bullet(T)_\Gamma$ such that $\lambda+\nu, \mu+\nu \in w(\mathbb{X}_\bullet(T)_\Gamma^+)$. It is well-known that the above complex of ℓ -adic vector spaces is non-zero only if $t_{\lambda+\nu} \leq t_{\mu+\nu}$ in Bruhat order, which is equivalent to $t_{w^{-1}(\lambda+\nu)} \leq t_{w^{-1}(\mu+\nu)}$, which is in turn equivalent to $w^{-1}(\lambda+\nu) \preceq w^{-1}(\mu+\nu)$ by Lemma 2.2, which is equivalent to $w^{-1}(\lambda) \preceq w^{-1}(\mu)$. The second statement follows from $R\mathrm{Hom}(J_\mu^w, J_\mu^w) \cong R\mathrm{Hom}(J_0^w, J_0^w) \cong \bar{\mathbb{Q}}_\ell$. \square

Lemma 7.8. *Let $\mathcal{F} \in D_B(\mathcal{F}\ell)$. Then for any $\mu \in \mathbb{X}_\bullet(T)_\Gamma$,*

$$H^*(\mathcal{F}\ell, J_\mu^w \star \mathcal{F}) \cong H^{*(w^{-1}(\mu), 2\rho)}(\mathcal{F}\ell, \mathcal{F}).$$

In particular, $H^(\mathcal{F}\ell, J_\mu^w) = H^{(w^{-1}(\mu), 2\rho)}(\mathcal{F}\ell, J_\mu^w) \cong \bar{\mathbb{Q}}_\ell$.*

Proof. Observe that for any $v \in \widetilde{W}$, $H^*(\mathcal{F}\ell, j_{v*} \star \mathcal{F}) \cong H^*(\mathcal{F}\ell, \mathcal{F})[\ell(v)]$. Indeed, let $C(v)$ be the Schubert cell in $\mathcal{F}\ell$ corresponding to v . Then we have $m : C(v) \tilde{\times} \mathcal{F}\ell \rightarrow \mathcal{F}\ell$ and $j_{v*} \star \mathcal{F} \cong m_*(\mathbb{Q}_\ell[\ell(v)] \tilde{\times} \mathcal{F})$. Now the lemma follows from the fact that $C(v) \tilde{\times} \mathcal{F}\ell \rightarrow \mathcal{F}\ell$ is an affine bundle over $\mathcal{F}\ell$. Therefore, for $\mu \in w(-\mathbb{X}_\bullet(T)_\Gamma^+)$, the lemma holds by the above fact and Lemma 2.1. The extension to all μ is immediate. \square

Let $W^w(\mathcal{F}\ell)$ be the full abelian subcategory of $P_B(\mathcal{F}\ell)$ generated by those $J_\mu^w, \mu \in \mathbb{X}_\bullet(T)_\Gamma$. Let $W^w(\mathcal{F}\ell)_{\succeq \mu}$ be the category of $W(\mathcal{F}\ell)$ generated by $J_\lambda^w, w^{-1}(\lambda) \succeq w^{-1}(\mu)$. For each object $\mathcal{F} \in W(\mathcal{F}\ell)$, we define a filtration

$$\mathcal{F} = \bigcup_{\mu} \mathcal{F}_{\succeq \mu}^w,$$

where $\mathcal{F}_{\succeq \mu}^w \in W^w(\mathcal{F}\ell)_{\succeq \mu}$ is the maximal subobject of \mathcal{F} belonging to $W^w(\mathcal{F}\ell)_{\succeq \mu}$. Then by Lemma 7.7,

$$\mathcal{F}_{\succeq \mu}^w / \bigcup_{w^{-1}(\mu') \succ w^{-1}(\mu)} \mathcal{F}_{\succeq \mu'}^w \cong J_{\mu}^w \otimes {}^w W_{\mathcal{F}}^{\mu},$$

where ${}^w W_{\mathcal{F}}^{\mu}$ is a finite dimensional \mathbb{Q}_{ℓ} vector space. A direct consequence of Lemma 7.8 is

Corollary 7.9. *Notations are as above. Then for any $\mathcal{F} \in W(\mathcal{F}\ell)$, we have*

$$H^*(\mathcal{F}\ell, \mathcal{F}) \cong \bigoplus_{\mu \in \mathbb{X}_{\bullet}(T)_{\Gamma}} H^*(\mathcal{F}\ell, J_{\mu}^w) \otimes {}^w W_{\mathcal{F}}^{\mu}.$$

Finally, let us prove Theorem 3.9. Let $\mu \in \mathbb{X}_{\bullet}(T)_{\Gamma}^+$. Let $\text{Supp}(\mu)$ denote the subset of \widetilde{W} consisting of those w such that $\mathcal{F}\ell_w \subset (\overline{\text{Gr}}_{\mathcal{G}, \mu})_{\tilde{0}}$. We need to show that $\text{Supp}(\mu) = \text{Adm}(\mu)$. We already know that $\text{Adm}(\mu) \subset \text{Supp}(\mu)$ (Lemma 3.8). By Proposition 7.6 and 7.5, we also know that the maximal elements in $\text{Supp}(\mu)$ (under the Bruhat order) belong to $\mathbb{X}_{\bullet}(T)_{\Gamma} \subset \widetilde{W}$. Let $t_{\mu'} \in \text{Supp}(\mu)$ be a maximal element. Then there exists some $w \in W_0$ such that $\mu' \in w(\mathbb{X}_{\bullet}(T)_{\Gamma}^+)$. By Proposition 7.6, $\mathcal{Z}_{\mu} \in W^w(\mathcal{F}\ell)$. Write $\mathcal{Z}_{\mu} = \cup_{\lambda} (\mathcal{Z}_{\mu})_{\succeq \lambda}^w$ so that the associated graded pieces are $J_{\lambda}^w \otimes {}^w W_{\mu}^{\lambda}$ as above (we write ${}^w W_{\lambda}^{\mu}$ instead of ${}^w W_{\mathcal{Z}_{\mu}}^{\lambda}$ for brevity). By Lemma 7.1, ${}^w W_{\mu}^{\mu'} \neq 0$. Also, being a maximal element in $\text{Supp}(\mu)$, $t_{\mu'}$ must have length $(\mu, 2\rho)$. Therefore, $(w^{-1}(\mu'), 2\rho) = (\mu, 2\rho)$. On the other hand, $t_{w(\mu)}$ is also a maximal element in $\text{Supp}(\mu)$ so that ${}^w W_{\mu}^{w(\mu)} \neq 0$. We claim that $\mu' = w(\mu)$. Otherwise, we would have

$$H^{(\mu, 2\rho)}(\mathcal{F}\ell, \mathcal{Z}_{\mu}) \supset {}^w W_{\mu}^{\mu'} \oplus {}^w W_{\mu}^{w(\mu)}$$

whose dimension would be at least two.

On the other hand, the map $f : \overline{\text{Gr}}_{\mathcal{G}, \mu} \rightarrow \tilde{C}$ is proper, and therefore $H^*(\mathcal{F}\ell, \mathcal{Z}_{\mu}) \cong \Psi_{\tilde{C}}(f_* \text{IC}_{\mu})$. Since $\overline{\text{Gr}}_{\mathcal{G}, \mu}|_{\tilde{C}} \cong \overline{\text{Gr}}_{\mu} \times \tilde{C}$, we have $H^*(\mathcal{F}\ell, \mathcal{Z}_{\mu}) \cong IH^*(\overline{\text{Gr}}_{\mu})$ where IH^* denotes the intersection cohomology of $\overline{\text{Gr}}_{\mu}$. It is well-known (for example see [MV]) that $IH^{(\mu, 2\rho)}(\overline{\text{Gr}}_{\mu}) \cong \mathbb{Q}_{\ell}$. Contradiction! Therefore, $\mu' = w(\mu)$. In other words, all the maximal elements on $\text{Supp}(\mu)$ are contained in $\text{Adm}(\mu)$. The theorem is proven.

Remark 7.2. One should be able to generalize [AB, Theorem 4] to the ramified case, which will imply Theorem 3.9 directly. Namely, $W^w(\mathcal{F}\ell)$ is indeed a monoidal abelian subcategory of $P_B(\mathcal{F}\ell)$ because $J_{\lambda}^w \star J_{\mu}^w \cong J_{\lambda+\mu}^w$. Let $\text{Gr}W^w(\mathcal{F}\ell)$ be the submonoidal category whose objects are direct sums of J_{λ}^w . Clearly, this category is equivalent to $\text{Rep}(\hat{T}^{\Gamma})$, where \hat{T} is the dual torus of T defined over $\bar{\mathbb{Q}}_{\ell}$, and \hat{T}^{Γ} is the Galois fixed subgroup. By taking the associated graded as the filtration of $\mathcal{F} \in W^w(\mathcal{F}\ell)$ defined before, one obtains a well-defined functor $\text{Gr} : W^w(\mathcal{F}\ell) \rightarrow \text{Gr}W^w(\mathcal{F}\ell)$. As explained in [AB, Lemma 16], this is a monoidal functor.

Since $\text{Gr}\mathcal{G}|_{\tilde{C}} \cong \text{Gr}_H \times \tilde{C}$, the nearby cycle functor indeed gives a monoidal functor from $\mathcal{Z} : P_{L+H}(\text{Gr}_H) \rightarrow W^w(\mathcal{F}\ell)$, where $P_{L+H}(\text{Gr}_H)$ is the category of L^+H -equivariant perverse sheaves on Gr_H , which is well-known to be equivalent to the category of representations of the Langlands dual group \hat{H} . One can use the similar argument proved by Gaitsgory in [Be, Appendix] to show that this functor is in fact central (see Section 2 of *loc. cit.* for the definition). Then by the same argument as [AB], one can show that $\text{Gr} \circ \mathcal{Z} : P_{L+H}(\text{Gr}_H) \rightarrow \text{Gr}W^w(\mathcal{F}\ell)$ is in fact a tensor

functor, which indeed equivalent to the restriction functor from the representations of \hat{H} to the representations of \hat{T}^Γ .

7.2. The monodromy. In this subsection, we determine the monodromy of the nearby cycle \mathcal{Z}_μ . This does not play a role for the coherence conjecture. But it is an important piece of the theory. We shall prove

Theorem 7.10. *The monodromy on \mathcal{Z}_μ is unipotent.*

In the case when G is split, this is again proved in [G]. However, the argument used in [G] used certain numerical results about the center of the affine Hecke algebra proved by Bernstein, which has not been written down explicitly in the literature for ramified groups⁷. On the other hand, there is another purely geometrical argument (see [AB, Appendix]), which can be applied directly to the non-split case. We record them here just for reader's convenience.

Now let us review the following general situation. Let X be a \mathbb{G}_m -variety. Let $a : \mathbb{G}_m \times X \rightarrow X$ be the action map. Then it makes sense to talk about \mathbb{G}_m -monodromic sheaves (cf. [Ve]). By definition, a sheaf (or perverse sheaf) \mathcal{F} on X is called monodromic if for every $r \in \mathbb{G}_m(k)$, $a_r^* \mathcal{F} \cong \mathcal{F}$, where $a_r : X \rightarrow X$ is $a_r(x) = a(r, x)$. A complex \mathcal{F} is called monodromic if all of its cohomology sheaves (or equivalently its perverse cohomology) are monodromic. If \mathcal{F} is a monodromic (perverse) sheaf, then it defines an action of the tame fundamental group $\mathbb{T} = \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$ of \mathbb{G}_m on \mathcal{F} (cf. [Ve, §5]), called the monodromic action. Let us briefly recall the construction of this action. Let μ_n denote the group of n th roots of unit so that $\mathbb{T} \cong \varprojlim_{p \nmid n} \mu_n$. First, we assume that \mathcal{F} is a monodromic sheaf with finite stalks. For $x \in X(k)$, let $a_x : \mathbb{G}_m \rightarrow X$ be $a_x(r) = a(r, x)$. Then it is easy to see that $a_x^* \mathcal{F}$ is locally constant on \mathbb{G}_m . By the argument of Proposition 3.2 of *loc. cit.*, $a_x^* \mathcal{F}$ is tame on \mathbb{G}_m . Now by constructibility, there exists some positive integer n such that $([n] \times \text{id})^* a^* \mathcal{F}$ is constant along the fibers of the projection $\text{pr} : \mathbb{G}_m \times X \rightarrow X$, where $[n] : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is given by $r \mapsto r^n$. For $r \in \mathbb{G}_m(k)$, let $s_r : X \rightarrow \mathbb{G}_m \times X, x \mapsto (r, x)$ be the section of pr . Then for an n th root of unit ζ_n ,

$$\mathcal{F} \cong s_1^*([n] \times \text{id})^* m^* \mathcal{F} \cong s_{\zeta_n}^*([n] \times \text{id})^* m^* \mathcal{F} \cong \mathcal{F}$$

defines an action of μ_n on \mathcal{F} . If \mathcal{F} has \mathbb{Z}_ℓ -coefficients, then one applies the above construction to the inverse system $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^n \mathbb{Z}_\ell$.

Remark 7.3. In *loc. cit.*, Verdier in fact worked in a more general situation. Namely, he defined and studied monodromic sheaves for cones (i.e. \mathbb{G}_m -invariant closed subschemes in \mathbb{A}_V^n where V is some base scheme). Our notion is a special case of his in the following sense. Let $j : \mathbb{G}_m \times X \rightarrow \mathbb{A}^1 \times X$ be the natural open embedding. Then if \mathcal{F} is a \mathbb{G}_m -monodromic sheaf in our sense, then $j_! a^* \mathcal{F}$ is a monodromic sheaf on \mathbb{A}_X^1 in Verdier's sense. In addition, for any \mathbb{G}_m -monodromic (perverse) sheaf \mathcal{F} on X , under the natural isomorphism $\text{End}(\mathcal{F}) \cong \text{End}(j_! a^* \mathcal{F})$, the \mathbb{T} -actions in the above sense is the same as the \mathbb{T} -action in Verdier's sense.

Now assume that $f : X \rightarrow \mathbb{A}^1$ be a \mathbb{G}_m -equivariant morphism, where \mathbb{G}_m acts on \mathbb{A}^1 by natural dilatation. Then $Y := f^{-1}(0)$ is naturally a \mathbb{G}_m -variety. Let \mathcal{F} be a \mathbb{G}_m -equivariant perverse sheaf on $X|_{\mathbb{G}_m}$, the following lemma can be deduced from [Ve, Proposition 7.1].

⁷Since the Satake isomorphism for ramified groups has been established in [HRo], maybe this can be done as well.

Lemma 7.11. *The nearby cycle $\Psi_X(\mathcal{F})$ is a \mathbb{G}_m -monodromic sheaf on Y . Let F_0 be the fractional field of the strict Henselian local ring of \mathbb{A}^1 at 0. Then the Galois action of $\text{Gal}(F_0^s/F_0)$ on $\Psi_X(\mathcal{F})$ (in the theory of nearby cycles) factors through the tame quotient, and it is equal to the opposite monodromic \mathbb{T} -action (defined above) on $\Psi_X(\mathcal{F})$.*

Proof. As in *loc. cit.*, one define $f^X : \mathbb{G}_m \times X \rightarrow \mathbb{A}^1$ given by $f^X(r, x) = rf(x)$. For \mathcal{F} a perverse sheaf on $X|_{\mathbb{G}_m}$, let $\mathcal{F}^X := \text{pr}^*\mathcal{F}$, where $\text{pr} : \mathbb{G}_m \times X \rightarrow X$ is the projection. In *loc. cit.*, it is proved that $j_!\Psi_{\mathbb{G}_m \times X}(\mathcal{F}^X)$ is \mathbb{G}_m -monodromic, where $j : \mathbb{G}_m \times Y \rightarrow \mathbb{A}^1 \times Y$ is the open immersion. In addition, (i) the Galois action of $\text{Gal}(F_0^s/F_0)$ on this sheaf (from the nearby cycle construction) factors through the tame quotient, and is equal to the opposite monodromic \mathbb{T} -action; (ii) under $s_1 : Y \rightarrow \mathbb{G}_m \times Y, s_1(y) = (1, y)$, $s_1^*\Psi_{\mathbb{G}_m \times X}(\mathcal{F}^X) \cong \Psi_X(\mathcal{F})^P$, where $P \subset \text{Gal}(F_0^s/F_0)$ is the wild inertial group.

In our situation, we have the action $a : \mathbb{G}_m \times X \rightarrow X$ and f is \mathbb{G}_m -equivariant so that $f^X = f \circ a$. In addition \mathcal{F} is \mathbb{G}_m -equivariant, $\mathcal{F}^X \cong a^*\mathcal{F}$. Since a is smooth, $\Psi_{\mathbb{G}_m \times X}(\mathcal{F}^X) \cong a^*\Psi_X(\mathcal{F})$ (compatible with the $\text{Gal}(F_0^s/F_0)$ -action), and therefore, $\Psi_X(\mathcal{F}) \cong s_1^*\Psi_{\mathbb{G}_m \times X}(\mathcal{F}^X) \cong \Psi_X(\mathcal{F})^P$. \square

To prove the theorem, we apply the above lemma to the case $\overline{\text{Gr}}_{\mathcal{G}, \mu} \rightarrow \tilde{C}$ thanks to §5.3. Then the theorem is a direct consequence of the following lemma. Let us endow $\mathcal{F}\ell$ with the \mathbb{G}_m -action from the isomorphism $\mathcal{F}\ell \cong (\overline{\text{Gr}}_{\mathcal{G}})_{\tilde{0}}$.

Lemma 7.12. *The object in $P_B(\mathcal{F}\ell)$ are \mathbb{G}_m -monodromic. In addition, the monodromic action is unipotent.*

Proof. By Lemma 5.5, the intersection cohomology sheaf of each Schubert variety is \mathbb{G}_m -equivariant. The lemma is clear. \square

8. APPENDIX I: LINE BUNDLES ON THE LOCAL MODELS FOR RAMIFIED UNITARY GROUPS

Since Theorem 1 is not quite identical to the original coherence conjecture given by Pappas and Rapoport, we explain here how to apply it to the local models. First, if the group G is of type A or C , we find that all the $a_i^\vee = 1$ in this case, and the formulation of Theorem 1 coincides with the original conjecture of Pappas and Rapoport. Namely, the central charge of $\mathcal{L}(\sum_{i \in Y} \epsilon_i)$ is $\sharp Y$. In fact, in these cases, it is proven in *loc. cit.* (using the result of [Go1, Go2, PR2]) that the coherence conjecture holds for μ being sum of minuscule coweights. In what follows, we mainly discuss the ramified unitary groups. Let us also mention that the orthogonal group cases can be discussed similarly. But this will be done on another occasion.

Let us change the notation in the main body of the paper to the following. Let \mathcal{O}_{F_0} be a completed discrete valuation ring with perfect residue field k with $\text{char } k \neq 2$ and fractional field F_0 . Let π_0 be the uniformizer. For example, $\mathcal{O} = k[[t]]$ with $\pi_0 = t$ as in the main body of the paper, or $\mathcal{O} = \mathbb{Z}_p^{ur}$, the completion of the maximal unramified extension of \mathbb{Z}_p and $\pi_0 = p$.

We will follow [PR4] (see also [PR3]). Let F/F_0 be a quadratic extension. Let (V, ϕ) be a split hermitian vector space over F of dimension ≥ 4 . That is, V is a vector space over F and ϕ is a hermitian form such that there is a basis e_1, \dots, e_n of V satisfying

$$\phi(e_i, e_{n+1-j}) = \delta_{ij}, \quad i, j = 1, \dots, n.$$

Let $G = \mathrm{GU}(V, \phi)$ be the group of unitary similitudes for (V, ϕ) , i.e. for any F_0 -algebra R ,

$$G(R) = \{g \in \mathrm{GL}(V \otimes_{F_0} R) \mid \phi(gv, gw) = c(g)\phi(v, w) \text{ for some } c(g) \in R^\times\}.$$

Then $G \otimes_{F_0} F \cong \mathrm{GL}_n \times \mathbb{G}_m$. The derived group G_{der} is the ramified special unitary group $\mathrm{SU}(V, \phi)$ consisting of those $g \in G(R)$ such that $\det(g) = c(g) = 1$.

We fix a square root π of π_0 . There are two associated F_0 -bilinear forms,

$$(v, w) = \mathrm{Tr}_{F/F_0}(\phi(v, w)), \quad \langle v, w \rangle = \mathrm{Tr}_{F/F_0}(\pi^{-1}\phi(v, w)).$$

Then $(-, -)$ is symmetric bilinear and $\langle -, - \rangle$ is alternating. For $i = 0, \dots, n-1$, set

$$\Lambda_i = \mathrm{span}_{\mathcal{O}_{\bar{F}}} \{\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n\},$$

and complete this into a selfdual periodic lattice chain by setting $\Lambda_{i+kn} = \pi^{-k}\Lambda_i$. Then $\langle -, - \rangle : \Lambda_{-j} \times \Lambda_j \rightarrow \mathcal{O}_{F_0}$ is a perfect pairing. In particular Λ_0 is self-dual for the alternating form $\langle -, - \rangle$.

Let us fix a minuscule coweight $\mu_{r,s}$ of G_F of signature (r, s) with $r \leq s, r+s = n$. That is

$$\mu_{r,s}(a) = (\mathrm{diag}\{a^{(s)}, 1^{(r)}\}, a)$$

where $a^{(s)}$ denotes s -copies of a . Let $E = F$ if $r \neq s$ and $E = F_0$ if $r = s$. Let $m = \lfloor \frac{n}{2} \rfloor$. Let $I \subset \{0, \dots, m\}$ be a non-empty subset with the requirement that if n is even and $m-1 \in I$, then $m \in I$ as well (see [PR4, §1.b] or [PR3, Remark 4.2.C] for the reason why we make this assumption).

Let us define the following moduli scheme $\mathcal{M}_I^{\mathrm{naive}}$ over \mathcal{O}_E . A point of $\mathcal{M}_I^{\mathrm{naive}}$ with values in an \mathcal{O}_E -scheme S is given by an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ -submodule $\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ for each $j \in \pm I + n\mathbb{Z}$ satisfying the following conditions:

- (1) as an \mathcal{O}_S -module, \mathcal{F}_j is locally on S a direct summand of rank n ;
- (2) for each $j < j', j, j' \in \pm I + n\mathbb{Z}$, the natural inclusion

$$\Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \rightarrow \Lambda_{j'} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$$

induces a morphism $\mathcal{F}_j \rightarrow \mathcal{F}_{j'}$, and the isomorphism $\pi : \Lambda_j \rightarrow \Lambda_{j-n}$ induces an isomorphism of \mathcal{F}_j with \mathcal{F}_{j-n} ;

- (3) under the perfect pairing induced by $\langle -, - \rangle : \Lambda_{-j} \times \Lambda_j \rightarrow \mathcal{O}_{F_0}$, $\mathcal{F}_{-j} = \mathcal{F}_j^\perp$, where \mathcal{F}_j^\perp is the orthogonal complement on \mathcal{F}_j ;
- (4) the determinant condition as in [PR4, §1.e.1, d)].

As explained in *loc. cit.*, for any I , $\mathcal{M}_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} E$ is isomorphic to the Grassmannian $\mathbb{G}(s, n)$ of s -planes in n -space. In addition, for $i \in I$, there is a natural projection $\mathcal{M}_I^{\mathrm{naive}} \rightarrow \mathcal{M}_{\{i\}}^{\mathrm{naive}}$ (if n is even and $i = m-1$, $\{i\}$ will mean $\{m-1, m\}$). Now the local model $\mathcal{M}_I^{\mathrm{loc}}$ is defined as the flat closure of the generic fiber $\mathcal{M}_I^{\mathrm{naive}} \otimes E$ inside $\mathcal{M}_I^{\mathrm{naive}}$.

The special fiber $\mathcal{M}_I^{\mathrm{naive}} \otimes k$ (and therefore $\mathcal{M}_I^{\mathrm{loc}} \otimes k$) embeds into the (partial) affine flag variety of the unitary group over $k((t))$. Namely, let (V', ϕ') be a split hermitian space over $k((u))$ ($u^2 = t$) with a standard basis e_1, \dots, e_n , such that $\phi'(e_i, e_{n+1-j}) = \delta_{ij}$. Let $\lambda_j, j \in \{0, 1, \dots, n-1\}$ be the standard lattices in V' defined similarly to Λ_j (replacing π by u and \mathcal{O}_F by $k[[u]]$ in the definition of Λ_j). For $I \subset \{0, \dots, m\}$ as before, write $I = i_0 < i_1 < \dots < i_k$ and let P_I be the group scheme over $k[[t]]$ which is the stabilizer of the lattice chain

$$\lambda_{i_0} \subset \dots \subset \lambda_{i_k} \subset u^{-1}\lambda_{i_0}$$

in $\mathrm{GU}(V', \phi')$. As explained in *loc. cit.*, this is not always a connected group scheme over $k[[t]]$. But if it is, then it is a parahoric group scheme of $\mathrm{GU}(V', \phi')$. In any case, the neutral connected component P_I^0 of P_I is a parahoric group scheme.

Consider the ind-scheme \mathcal{F}_I which to a k -algebra R associates the set of sequences of $R[[u]]$ -lattice chains

$$L_{i_0} \subset \cdots \subset L_{i_k} \subset u^{-1}L_{i_0}$$

in $V' \otimes_{k((u))} R((u))$ together with an $R[[t]]$ -lattice $L \subset R((t))$ satisfying conditions a) and b) as in [PR4, §3.b] (observe that we replace $\alpha \in R((t))^\times / R[[t]]^\times$ in *loc. cit.* by a lattice $L \subset R((t))$, which seems more natural). Then

$$\mathcal{F}_I \cong \mathrm{LGU}(V', \phi') / L^+ P_I$$

and $\mathrm{LGU}(V', \phi') / L^+ P_I^0$ is either isomorphic to $\mathrm{LGU}(V', \phi') / L^+ P_I$ or to the disjoint union of two copies of $\mathrm{LGU}(V', \phi') / L^+ P_I$. In addition, for such I , one can canonically associate to it a subset $Y \subset \mathbf{S}$ (\mathbf{S} are the set of vertices in the local Dynkin diagram of $\mathrm{GU}(V', \phi')$) such that $\mathcal{F}_I^Y = \mathrm{LGU}(V', \phi') / L^+ P_I^0$. Indeed, by [PR3, Remark 10.3] (see also [PR4, §1.2.3]), one can identify \mathbf{S} with $\{0, 1, \dots, m\}$, if $n = 2m + 1$, resp. $\{0, 1, \dots, m - 2, m, m'\}$, if $n = 2m$, where m' is a formal symbol as defined in [PR3, §4], to which a lattice of V'

$$\lambda_{m'} = \mathrm{span}_{k[[u]]} \{u^{-1}e_1, \dots, u^{-1}e_{m-1}, e_m, u^{-1}e_{m+1}, e_{m+2}, \dots, e_{2m}\}$$

is associated. Then $Y = I$ in all cases except when $n = 2m$, $\{m - 1, m\} \subset I$, in which case $Y = (I \setminus \{m - 1\}) \cup \{m'\}$.

Remark 8.1. (i) Observe that if $n = 2m + 1$, under our identification of $\{0, 1, \dots, m\}$ with \mathbf{S} (the set of vertices of the local Dynkin diagram), i goes to the label $m - i$ in Kac's book ([Kac, p.p. 55]), and if $n = 2m$, under the identification of $\{0, \dots, m - 2, m, m'\}$ with \mathbf{S} , i goes to $m - i$ for $i \leq m - 2$ and $\{m, m'\}$ go to $\{0, 1\}$.

(ii) As pointed out in [PR3, PR4], if $n = 2m + 1$, then $P_{\{0\}}$ and $P_{\{m\}}$ are the special parahoric group schemes, and if $n = 2m$, then $P_{\{m\}}, P_{\{m'\}}$ are the special parahoric group schemes. We further point out: (1) let $n = 2m + 1$. Then (a) $P_{\{0\}}$ is the special parahoric determined by a pinning of $\mathrm{GL}_{2n+1} \times \mathbb{G}_m$, i.e. the group scheme \mathcal{G}_{v_0} as in (2.1.2), and its reductive quotient is GO_{2m+1} ; and (b) the special parahoric $P_{\{m\}}$ has reductive quotient GSp_{2m} , but it is not of the form (2.1.2). (2) Let $n = 2m$. Then both $P_{\{m\}}, P_{\{m'\}}$ are of the form (2.1.2), and their reductive quotients are both isomorphic to GSp_{2m} .

Fix the isomorphisms $\Lambda_j \otimes_{k[[t]]} k \cong \lambda_j \otimes k$, compatible with the actions of π and u , by sending $e_i \rightarrow e_i$. Now we embed the special fiber $\mathcal{M}_I^{\mathrm{naive}} \otimes k$ into \mathcal{F}_I as follows: for every k -algebra R ,

$$\mathcal{F}_j \subset (\Lambda_j \otimes k) \otimes_k R \cong (\lambda_j \otimes k) \otimes_k R,$$

and let $L_j \subset \lambda_j \otimes R[[t]]$ be the inverse image of \mathcal{F}_j under $\lambda_j \otimes R[[t]] \rightarrow \lambda_j \otimes \mathcal{O}_S$. In addition, let $L = t^{-1}R[[t]] \subset R((t))$. This gives the embedding

$$\iota_I : \mathcal{M}_I^{\mathrm{naive}} \otimes k \rightarrow \mathcal{F}_I.$$

It is proved in [PR4, Proposition 3.1] that $\mathcal{A}^I(\mu_{r,s})$ is contained in $\mathcal{M}_I^{\mathrm{loc}} \otimes_{\mathcal{O}_E} k$ under ι_I .

One of the main theorems in [PR4, Theorem 0.1] (under the assumption of the coherence conjecture) is

Theorem 8.1. $\mathcal{A}^I(\mu_{r,s}) = \mathcal{M}_I^{\text{loc}} \otimes_{\mathcal{O}_E} k$. Therefore, the special fiber of $\mathcal{M}_I^{\text{loc}}$ is reduced and each irreducible component is normal, Cohen-Macaulay and Frobenius-split.

To prove it, one needs to construct a natural line bundle on $\mathcal{M}_I^{\text{naive}}$ and apply the coherence conjecture to compare the dimensions of H^0 of this line bundle over the generic and the special fibers. There are several choices of the natural line bundles. One of them will be given in [PZ], after we give a group theoretical description of $\mathcal{M}_I^{\text{naive}}$. Here, we follow the originally approach of [PR3, PR4] to construct another line bundle \mathcal{L}_I , which is more down to earth.

First, if $I = \{j\}$, we define $\mathcal{L}_{\{j\}}$ over $\mathcal{M}_{\{j\}}^{\text{naive}}$ whose value at the \mathcal{O}_S -point given by $\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_E} \mathcal{O}_S$ is $\det(\mathcal{F}_j)^{-1}$. If $n = 2m$, we also define $\mathcal{L}_{\{m-1,m\}}$ over $\mathcal{M}_{\{m-1,m\}}^{\text{naive}}$ whose value at the \mathcal{O}_S -point of given by $\mathcal{F}_{m-1} \subset \mathcal{F}_m$ is $\det(\mathcal{F}_{m-1})^{-1} \otimes \det(\mathcal{F}_m)^{-1}$. For general I , the line bundle \mathcal{L}_I is defined as the tensor product of those $\mathcal{L}_{\{j\}}$ or $\mathcal{L}_{\{m-1,m\}}$ along all possible projections $\mathcal{M}_I^{\text{naive}} \rightarrow \mathcal{M}_j^{\text{naive}}$ or $\mathcal{M}_I^{\text{naive}} \rightarrow \mathcal{M}_{\{m-1,m\}}^{\text{naive}}$.

The restriction of $\mathcal{L}_{\{j\}}$ to the generic fiber $\mathcal{M}_{\{j\}}^{\text{naive}} \otimes_E F \cong \mathbb{G}(s, n)$ is isomorphic to $\mathcal{L}_{\det}^{\otimes 2}$, where \mathcal{L}_{\det} is the determinant line bundle on $\mathbb{G}(s, n)$, which is the positive generator of the Picard group of $\mathbb{G}(s, n)$. On the other hand, the restriction of $\mathcal{L}_{\{m-1,m\}}$ to the generic fiber of $\mathcal{M}_{\{m-1,m\}}^{\text{naive}}$ is isomorphic to $\mathcal{L}_{\det}^{\otimes 4}$.

Proposition 8.2. Under the canonical isomorphism $\mathcal{A}^I(\mu_{r,s})^\circ \cong \mathcal{A}^I(\mu_{r,s})$, the line bundle \mathcal{L}_I , when restricted to $\mathcal{A}^I(\mu_{r,s})$ is isomorphic to the restriction of $\mathcal{L}(\sum_{j \in Y} \kappa(j)\epsilon_j)$ to $\mathcal{A}^I(\mu_{r,s})^\circ$, where

- (1) if $n = 2m + 1$, then $\kappa(j) = 1$ for $j = 0, 1, \dots, m-1$ and $\kappa(m) = 2$;
- (2) if $n = 2m$, then $\kappa(j) = 1$ for $j = 0, \dots, m-2$ and $\kappa(m) = \kappa(m') = 2$.

Proof. Let us first introduce a convention. In what follows, when we write λ_j , we consider it as a $k[[u]]$ -lattice. If we just remember its $k[[t]]$ -lattice structure, we denote it by $\lambda_j/k[[t]]$.

Clearly, we can assume that $I = \{j\}$ or when $n = 2m$ we shall also consider $I = \{m-1, m\}$. The latter case will be treated at the end of the proof. So we first assume that $j \neq m-1$.

Observe that we have a natural closed embedding of ind-schemes

$$LGU(V', \phi')/L^+P_{\{j\}} \cong \mathcal{F}_{\{j\}} \rightarrow \text{Gr}_{\text{GL}(\lambda_j)} \times \text{Gr}_{\mathbb{G}_m}$$

by just remembering the lattices $L_j \subset \lambda_j \otimes_{k[[u]]} R((u))$ and $L \subset R((t))$. By definition, the line bundle $\iota_{\{j\}}^* \mathcal{L}_{\{j\}}$ on $\mathcal{M}_{\{j\}}^{\text{naive}} \otimes_{\mathcal{O}_E} k$ is the pullback of the determinant line bundle on $\text{Gr}_{\text{GL}(\lambda_j)}$ along the above map.

Let $\text{SU}(V', \phi')$ be the special unitary group. As explained in [PR3, §4], $P'_I = P_I \cap \text{SU}(V', \phi')$ is a parahoric group scheme of $\text{SU}(V', \phi')$. By [PR3, §6], we have

$$\begin{array}{ccc} \text{LSU}(V', \phi')/L^+P'_{\{j\}} & \longrightarrow & LGU(V', \phi')/L^+P_{\{j\}} \\ \downarrow & & \downarrow \\ \text{Gr}_{\text{SL}(\lambda_j)} & \longrightarrow & \text{Gr}_{\text{GL}(\lambda_j)} \end{array}$$

where the ind-schemes in the left column are identified with the reduced part of neutral connected components of the ind-schemes in the right column. Since the isomorphism $\mathcal{A}^I(\mu_{r,s})^\circ \cong \mathcal{A}^I(\mu_{r,s})$ is obtained from the translation by some $g \in \text{GU}(V', \phi')(F)$, it is enough to prove

Lemma 8.3. The pullback of \mathcal{L}_{\det} along $\text{LSU}(V', \phi')/L^+P'_{\{j\}} \rightarrow \text{Gr}_{\text{SL}(\lambda_j)}$ is $\mathcal{L}(\kappa(j)\epsilon_j)$.

Proof. Assume that $j \neq 0, m$, and in the case $n = 2m$, $j \neq m - 1$. Consider the rational line $\mathbb{P}_j^1 \subset \mathrm{LSU}(V', \phi')/L^+P'_{\{j\}}$ given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$ -family of lattices $L_{j,s} = u^{-1}k[[u]]e_1 + \cdots + u^{-1}k[[u]]e_{j-1} + u^{-1}k[[u]](e_j + se_{j+1}) + k[[u]]e_{j+1} + \cdots + k[[u]]e_n$. It is easy to see that the restriction of \mathcal{L}_{\det} to this rational line is $\mathcal{O}(1)$. In fact, by the map

$$L_{j,s} \rightarrow L_{j,s}/\left(\sum_{r \leq j-1} u^{-1}k[[u]]e_r + \sum_{r \geq j} k[[u]]e_r\right),$$

this rational curve \mathbb{P}_j^1 is identified with the $\mathrm{Gr}(1, 2)$ classifying lines in the 2-dimensional k -vector space generated by $\{u^{-1}e_j, u^{-1}e_{j+1}\}$ and clearly the restriction of the determinant line bundle of $\mathrm{Gr}_{\mathrm{SL}(\lambda_j)}$ is the determinant line bundle on $\mathrm{Gr}(1, 2)$. Therefore, $\kappa(j) = 1$ if $j \neq 0, m$ (and $j \neq m - 1$ if $n = 2m$).

If $j = 0$, consider the rational line $\mathbb{P}_0^1 \subset \mathrm{LSU}(V', \phi')/L^+P'_{\{0\}}$ given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$ -family of lattices

$$(8.0.1) \quad L_s = k[[u]]e_1 + \cdots + k[[u]]e_{n-1} + k[[u]](e_n + su^{-1}e_1).$$

By the same reasoning as above, the restriction of \mathcal{L}_{\det} to this rational line is $\mathcal{O}(1)$. Therefore, $\kappa(0) = 1$.

Now, if $n = 2m + 1$ and $j = m$ or $n = 2m$ and $j = m$ or m' , we will prove that $2 \mid \kappa(j)$. Assuming this, to prove the lemma it is enough to find some rational line $\mathbb{P}_j^1 \subset \mathrm{LSU}(V', \phi')/L^+P'_{\{j\}}$ such that the restriction of \mathcal{L}_{\det} to it is $\mathcal{O}(2)$. If $n = 2m + 1$, we can take the rational line \mathbb{P}_m^1 given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$ -family of lattices

$$L_s = u^{-1}k[[u]]e_1 + \cdots + u^{-1}k[[u]]e_{m-1} + u^{-1}k[[u]](e_m + se_{m+1} - \frac{s^2}{2}e_{m+2}) + k[[u]]e_{m+1} + \cdots + k[[u]]e_n.$$

To see that \mathcal{L}_{\det} restricts to $\mathcal{O}(2)$, consider the map

$$L_s \rightarrow L_s/\left(\sum_{r \leq m-1} u^{-1}k[[u]]e_r + \sum_{r \geq m} k[[u]]e_r\right),$$

which gives rise to embeddings, $\mathbb{P}_m^1 \subset \mathrm{Gr}(1, 3) \subset \mathrm{Gr}_{\mathrm{SL}(\lambda_m)}$. Here $\mathrm{Gr}(1, 3)$ classifies lines in the 3-dimensional k -vector space generated by $\{u^{-1}e_m, u^{-1}e_{m+1}, u^{-1}e_{m+2}\}$. Clearly, the pullback of \mathcal{L}_{\det} along $\mathrm{Gr}(1, 3) \rightarrow \mathrm{Gr}_{\mathrm{SL}(\lambda_m)}$ is the determinant line bundle and the embedding $\mathbb{P}_m^1 \rightarrow \mathrm{Gr}(1, 3)$ is quadratic, the claim follows.

If $n = 2m$ and $j = m$ ($j = m'$ case is the same), we can take the rational line \mathbb{P}_m^1 given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$ -family of lattices

$$L_s = u^{-1}k[[u]]e_1 + \cdots + u^{-1}k[[u]]e_{m-2} + u^{-1}k[[u]](e_{m-1} + se_{m+1}) + u^{-1}k[[u]](e_m - se_{m+2}) + k[[u]]e_{m+1} + \cdots + k[[u]]e_n.$$

To see that \mathcal{L}_{\det} restricts to $\mathcal{O}(2)$, consider the map

$$L_s \rightarrow L_s/\left(\sum_{r \leq m-2} u^{-1}k[[u]]e_r + \sum_{r \geq m-1} k[[u]]e_r\right),$$

which gives rise to embeddings, $\mathbb{P}_m^1 \subset \mathrm{Gr}(2, 4) \subset \mathrm{Gr}_{\mathrm{SL}(\lambda_m)}$. Here $\mathrm{Gr}(2, 4)$ classifies planes in the 4-dimensional k vector space generated by $\{u^{-1}e_{m-1}, \dots, u^{-1}e_{m+2}\}$. The restriction of \mathcal{L}_{\det} to $\mathrm{Gr}(2, 4)$ is the determinant line bundle, and therefore it is enough to see that the restriction of the determinant line bundle on $\mathrm{Gr}(2, 4)$ along $\mathbb{P}_m^1 \rightarrow \mathrm{Gr}(2, 4)$ is $\mathcal{O}(2)$. We use the determinant line bundle on $\mathrm{Gr}(2, 4)$ to embed

$\text{Gr}(2, 4)$ into $\mathbb{P}(V)$, where V is generated by $\{\{u^{-1}e_i \wedge u^{-1}e_j \mid m-1 \leq i < j \leq m+2\}$, then the composition $\text{Spec}k[s] \subset \mathbb{P}_m^1 \rightarrow \text{Gr}(2, 4) \rightarrow \mathbb{P}(V) \setminus \{u^{-1}e_{m-1} \wedge u^{-1}e_m\}$ is given by

$$s \mapsto su^{-1}e_{m-1} \wedge u^{-1}e_{m+2} + su^{-1}e_m \wedge u^{-1}e_{m+1} - s^2u^{-1}e_{m+1} \wedge u^{-1}e_{m+2}.$$

The claim is clear from this description.

So it remains to prove $2 \mid \kappa(j)$ for $n = 2m + 1, j = m$, or $n = 2m, j = m$ or m' . Recall that when regarding V' as a vector space over $k((t))$, it has a split symmetric bilinear form

$$(v, w) = \text{Tr}_{k((u))/((t))}(\phi'(v, w)).$$

Observe that when $n = 2m + 1, j = m$, or $n = 2m, j = m$ or m' , $\lambda_j/k[[t]]$ is a Lagrangian, i.e. $\lambda_j \subset \widehat{\lambda}_j^s$ and $\dim_k(\widehat{\lambda}_j^s/\lambda_j) = 0$ or 1 , where

$$\widehat{\lambda}_j^s = \{v \in V' \mid (v, \lambda_j) \subset \mathcal{O}\}.$$

Let $\text{Lag}(V') \subset \text{Gr}_{\text{SL}(\lambda_j/k[[t]])}$ denote the subspace of Lagrangian lattices in V' . Then the morphism

$$\text{LSU}(V', \phi')/L^+P'_{\{j\}} \rightarrow \text{Gr}_{\text{SL}(\lambda_j)} \rightarrow \text{Gr}_{\text{SL}(\lambda_j/k[[t]])}$$

factors through

$$\text{LSU}(V', \phi')/L^+P'_{\{j\}} \rightarrow \text{Lag}(V') \rightarrow \text{Gr}_{\text{SL}(\lambda_j/k[[t]])}.$$

It is by definition that the pullback of \mathcal{L}_{\det} along $\text{Gr}_{\text{SL}(\lambda_j)} \rightarrow \text{Gr}_{\text{SL}(\lambda_j/k[[t]])}$ is \mathcal{L}_{\det} , and it is well-known (for example, see [BD, §4]) that the pullback of \mathcal{L}_{\det} along $\text{Lag}(V') \rightarrow \text{Gr}_{\text{SL}(\lambda_j/k[[t]])}$ admits a square root (the Pfaffian line bundle). The lemma follows. \square

To deal with the case $n = 2m$ and $I = \{m-1, m\}$, observe there is a map

$$\text{LGU}(V', \phi')/L^+P_I \rightarrow \text{Gr}_{\text{GL}(\lambda_m)} \times \text{Gr}_{\text{GL}(\lambda_{m'})}$$

by sending $L_{m-1} \subset L_m$ to L_m, gL_m , where g is the unitary transform $e_m \mapsto e_{m+1}, e_{m+1} \mapsto e_m$ and $e_i \mapsto e_i$ for $i \neq m, m+1$. One observes that $\iota_I^* \mathcal{L}_I$ on $\mathcal{M}_I^{\text{naive}} \otimes_{\mathcal{O}_E} k$ is the pullback along the above map of the tensor product of the determinant line bundles (on each factor). \square

Finally, let us see why this proposition can be used to deduce Theorem 0.1 of [PR4]. First let a_i^\vee be the Kac labeling as in [Kac, §6.1]. Using Remark 8.1 (i), by checking all the cases, we find that $a_i^\vee \kappa(i) = 2$. Let \mathcal{L}_I be the line bundle on $\mathcal{M}_I^{\text{loc}}$. Then for $a \gg 0$,

$$\dim \Gamma(\mathcal{M}_I^{\text{loc}} \otimes_{\mathcal{O}_E} k, \mathcal{L}_I^a) = \dim \Gamma(\mathcal{M}_I^{\text{loc}} \otimes_{\mathcal{O}_E} E, \mathcal{L}_I^a).$$

By the above proposition and [PR4, Proposition 3.1], the left hand side

$$\dim \Gamma(\mathcal{M}_I^{\text{loc}} \otimes_{\mathcal{O}_E} k, \mathcal{L}_I^a) \geq \dim \Gamma(\mathcal{A}^I(\mu_{r,s})^\circ, \mathcal{L}(a \sum_{i \in Y} \kappa(i) \epsilon_i)),$$

and the central charge of $\mathcal{L}(a \sum_{i \in Y} \kappa(i) \epsilon_i)$ is

$$\sum_{i \in Y} a a_i^\vee \kappa(i) = 2a \sharp I.$$

The line bundle on right hand side is just the $2a \sharp I$ -power of the ample generator of the Picard group of $\mathbb{G}(s, n)$. Then by Theorem 1, Theorem 8.1 holds.

9. APPENDIX II: SOME RECOLLECTIONS AND PROOFS

We collect some algebro-geometrical results needed in the main body of the paper.

9.1. Deformation to the normal cone. Let C be a smooth curve over an algebraically closed field k . Let \mathcal{X} be a scheme faithfully flat and (affine) over C . Let $x \in C(k)$ be a point and let \mathcal{X}_x denote the fiber of \mathcal{X} over x . Let $Z \subset \mathcal{X}_x$ be a closed subscheme. Consider the following functor \mathcal{X}_Z on the category of flat C -schemes: for each $V \rightarrow C$,

$$\mathcal{X}_Z(V) = \{f \in \text{Hom}_C(V, \mathcal{X}) \mid f_x : V_x \rightarrow \mathcal{X}_x \text{ factors through } V_x \rightarrow Z \subset \mathcal{X}_x\}.$$

It is well known that this functor is represented by a scheme (affine) and flat over C , usually called the deformation to the normal cone (or called the dilatation of Z on \mathcal{X} , see [BLR, §3.2]). Indeed, the construction is easy if \mathcal{X} is affine over C . Namely, we can assume that C is affine and x is defined by a local parameter t . Assume that \mathcal{A} be the \mathcal{O}_C -algebra defining \mathcal{X} over C , and let $\mathcal{I} \subset \mathcal{A}$ be the ideal defining $Z \subset \mathcal{X}$. Then $t\mathcal{A} \subset \mathcal{I}$ and let $\mathcal{B} = \mathcal{A}[\frac{i}{t}, i \in \mathcal{I}] \subset \mathcal{A}[t^{-1}]$. It is easy to see that \mathcal{B} is flat over \mathcal{O}_C and $\text{Spec} \mathcal{B}$ represents \mathcal{X}_Z .

There is a natural morphism $\mathcal{X}_Z \rightarrow \mathcal{X}$ which induces an isomorphism over $C - \{x\}$ and over x it factors as $(\mathcal{X}_Z)_x \rightarrow Z \rightarrow \mathcal{X}_x$. If \mathcal{X} is smooth over C , and Z is a smooth closed subscheme of \mathcal{X}_x , then \mathcal{X}_Z is also smooth over C . Indeed, étale locally on \mathcal{X}_x , the map $(\mathcal{X}_Z)_x \rightarrow Z$ can be identified with the map from the normal bundle of Z inside \mathcal{X}_x to Z , which justifies the name of the construction.

Now let \mathcal{G}_1 be a connected affine smooth group scheme over the curve C . Let $x \in C(k)$ and let $(\mathcal{G}_1)_x$ be the fiber of \mathcal{G}_1 at x . Let $P \subset (\mathcal{G}_1)_x$ be a smooth closed subgroup. Let $\mathcal{G}_2 = (\mathcal{G}_1)_P$. This is indeed a smooth connected affine group scheme over C . By restriction to x , we have $r : \text{Bun}_{\mathcal{G}_2} \rightarrow \mathbb{B}(\mathcal{G}_2)_x$ and $r : \text{Bun}_{\mathcal{G}_1} \rightarrow \mathbb{B}(\mathcal{G}_1)_x$ (here we assume that C is a complete curve).

Proposition 9.1. *We have the following Cartesian diagram*

$$\begin{array}{ccccc} \text{Bun}_{\mathcal{G}_2} & \xrightarrow{r} & \mathbb{B}(\mathcal{G}_2)_x & \longrightarrow & \mathbb{B}P \\ \downarrow & & & & \downarrow \\ \text{Bun}_{\mathcal{G}_1} & \xrightarrow{r} & \mathbb{B}(\mathcal{G}_1)_x & & \end{array}$$

Proof. Let $\text{Spec} R$ be a noetherian⁸ affine scheme. Let \mathcal{E} be a \mathcal{G}_1 -torsor on C_R and \mathcal{E}_P be a P -torsor on V together with an isomorphism $\mathcal{E}_P \times^P (\mathcal{G}_1)_x \cong \mathcal{E}|_{\{x\} \times \text{Spec} R}$. We need to construct a \mathcal{G}_2 -torsor \mathcal{E}' satisfying obvious conditions.

As a scheme over C , \mathcal{E} is faithfully flat. Its fiber over x is $\mathcal{E}|_{\{x\} \times \text{Spec} R}$. Let Z be the closed subscheme of \mathcal{E}_x given by the closed embedding

$$\mathcal{E}_P \subset \mathcal{E}_P \times^P (\mathcal{G}_1)_x \cong \mathcal{E}|_{\{x\} \times \text{Spec} R}.$$

Then \mathcal{E}_Z is a scheme affine and flat over C , together with a morphism $\mathcal{E}_Z \rightarrow \mathcal{E}$. Therefore, \mathcal{E}_Z is a scheme over C_R . We claim that \mathcal{E}_Z is a \mathcal{G}_2 -torsor over C_R . First, \mathcal{E}_Z is faithfully flat over C_R . Indeed, by the local criterion of flatness, it is enough to prove that $\mathcal{E}_Z|_{\{x\} \times \text{Spec} R}$ is faithfully flat over $\text{Spec} R$. But this is clear, since étale locally on $\mathcal{E}|_{\{x\} \times \text{Spec} R}$, there is an isomorphism between $\mathcal{E}_Z|_{\{x\} \times \text{Spec} R}$ and the normal bundle of $\mathcal{E}_P \subset \mathcal{E}_P \times^P (\mathcal{G}_1)_x$. Next, there is an action of \mathcal{G}_2 on \mathcal{E}_Z . Indeed,

⁸This does not lose any generality since all the stacks are locally of finite presentation.

the map $\mathcal{E}_Z \times_{C_R} \mathcal{G}_2 \mapsto \mathcal{E} \times_{C_R} \mathcal{G}_1 \rightarrow \mathcal{E}$, when restricted to the fiber over x , factors through Z . Therefore, by the definition of \mathcal{E}_Z , it gives rise to a map

$$\mathcal{E}_Z \times_{C_R} \mathcal{G}_2 \mapsto \mathcal{E}_Z.$$

Finally, it is easy to see that

$$\mathcal{E}_Z \times_{C_R} \mathcal{E}_Z \cong \mathcal{E}_Z \times_{C_R} \mathcal{G}_2.$$

Indeed, the l.h.s represents the scheme $(\mathcal{E} \times_{C_R} \mathcal{E})_{Z \times_{\text{Spec } R} Z}$ and the r.h.s represents the scheme $(\mathcal{E} \times_{C_R} \mathcal{G}_1)_{Z \times_{\text{Spec } R} P}$. Then the desired isomorphism follows from

$$(\mathcal{E} \times_{C_R} \mathcal{E})_{Z \times_{\text{Spec } R} Z} \cong (\mathcal{E} \times_{C_R} \mathcal{G}_1)_{Z \times_{\text{Spec } R} P}.$$

□

9.2. Frobenius morphisms. Let us review some basic facts about the Frobenius morphisms for a variety X over an algebraically closed field of characteristic $p > 0$. First assume that X is smooth and let ω_X be its canonical sheaf. Then there is the following isomorphism

$$(9.2.1) \quad \mathfrak{D} : F_* \omega_X^{1-p} \cong \text{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X),$$

where $F : X \rightarrow X$ is the absolute Frobenius map of X and ω_X is the canonical sheaf of X . Let $x \in X$ be a closed point and x_1, \dots, x_n be a sequence of regular parameters. Then locally around x (i.e. at the completed local ring $\hat{\mathcal{O}}_{X,x}$), the above isomorphism is given by

$$(9.2.2) \quad \begin{aligned} & \mathfrak{D}(x_1^{m_1} \cdots x_n^{m_n} (dx_1 \cdots dx_n)^{1-p}) (x_1^{\ell_1} \cdots x_n^{\ell_n}) \\ &= \begin{cases} 0 & \text{if } p \nmid m_i + \ell_i + 1 \text{ for some } i \\ v_1^{(m_1 + \ell_1 - p + 1)/p} \cdots v_n^{(m_n + \ell_n - p + 1)/p} & \text{otherwise} \end{cases} \end{aligned}$$

Next, assume that X is normal. It is still make sense to talk about the canonical sheaf ω_X and its any n th power $\omega_X^{[n]}$. Namely, let $j : X^{sm} \rightarrow X$ be the open immersion of its smooth part. Then by definition $\omega_X^{[n]} := j_* \omega_{X^{sm}}^n$. The isomorphism (9.2.1) still holds in this situation. Observe that there is a natural map $(\omega_X^{[\pm 1]})^{\otimes n} \rightarrow \omega_X^{[\pm n]}$ ($n > 0$) which is not necessarily an isomorphism. In what follows, we use ω_X^n to denote $\omega_X^{[n]}$ if no confusion will rise. Let us recall that if in addition X is Cohen-Macaulay, ω_X is the dualizing sheaf.

Next, we consider a flat family $f : X \rightarrow V$ of varieties which is fiberwise normal and Cohen-Macaulay. In addition, let us assume that V is smooth, so that the total space X is also normal and Cohen-Macaulay. In this case, the relative dualizing sheaf $\omega_{X/V}$ commutes with base change and is flat over V . We have $\omega_X \cong f^* \omega_V \otimes \omega_{X/V}$. Let $X^{(p)}$ be the Frobenius twist of X over V , i.e. the pullback of X along the absolute Frobenius endomorphism $F : V \rightarrow V$. Let $F_{X/V} : X \rightarrow X^{(p)}$ be the relative Frobenius morphism, and let $\varphi : X^{(p)} \rightarrow X$ be the map such that the composition $\varphi \circ F_{X/V}$ is the absolute Frobenius morphism F for X . Then

$$(9.2.3) \quad \mathfrak{R}\mathfrak{D} : (F_{X/V})_* \omega_{X/V}^{1-p} \cong \text{Hom}_{\mathcal{O}_{X^{(p)}}}((F_{X/V})_* \mathcal{O}_X, \mathcal{O}_{X^{(p)}}).$$

Here $\omega_{X/V}^n$, as in the absolute case, is the pushout of the n th tensor power of the relative canonical sheaf on $X^{rel,sm}$, the maximal open part of X such that $f|_{X^{rel,sm}}$ is smooth. In addition, we have the follows morphisms

$$(9.2.4) \quad f^* F_* \omega_V^{1-p} \cong f^* \text{Hom}(F_* \mathcal{O}_V, \mathcal{O}_V) \cong \text{Hom}(\varphi_* \mathcal{O}_{X^{(p)}}, \mathcal{O}_X),$$

$$(9.2.5) \quad F_*\omega_{X/V}^{1-p} \otimes f^*F_*\omega_V^{1-p} \cong F_*\omega_{X/V}^{1-p} \otimes F_*f^*\omega_V^{1-p} \rightarrow F_*\omega_X^{1-p}.$$

The morphisms (9.2.1), (9.2.3)-(9.2.5) fit into the following commutative diagram (9.2.6)

$$\begin{array}{ccc} \varphi_*\mathcal{H}om((F_{X/V})_*\mathcal{O}_X, \mathcal{O}_{X^{(p)}}) \otimes \mathcal{H}om(\varphi_*\mathcal{O}_{X^{(p)}}, \mathcal{O}_X) & \longrightarrow & \mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X) \\ \uparrow & & \uparrow \\ F_*\omega_{X/V}^{1-p} & \otimes & f^*F_*\omega_V^{1-p} \longrightarrow F_*\omega_X^{1-p}. \end{array}$$

Finally, let W be another smooth variety over k and let $g : W \rightarrow V$ be a k -morphism (not necessarily flat). By abuse of the notation, we still use g to denote the base change maps $X_W \rightarrow X$ and $(X_W)^{(p)} \cong X_W^{(p)} \rightarrow X^{(p)}$. Then the following diagram is commutative.

$$(9.2.7) \quad \begin{array}{ccc} g^*(F_{X/V})_*\omega_{X/V}^{1-p} & \xrightarrow{\cong} & g^*\mathcal{H}om((F_{X/V})_*\mathcal{O}_X, \mathcal{O}_{X^{(p)}}) \\ \cong \downarrow & & \downarrow \cong \\ (F_{X_W/W})_*\omega_{X_W/W}^{1-p} & \xrightarrow{\cong} & \mathcal{H}om((F_{X_W/W})_*\mathcal{O}_{X_W}, \mathcal{O}_{X_W^{(p)}}). \end{array}$$

To prove (9.2.3), (9.2.6) and (9.2.7), one can first assume that X is smooth over V and then the assertions are easy consequences of the relative duality theorem. To extend it to the flat family with fiberwise normal and Cohen-Macaulay case is immediate.

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XINWEN ZHU: DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, ONE OXFORD STREET, CAMBRIDGE, MA 02138, USA

E-mail address: xinwenz@math.harvard.edu